# Valuation of fixed and variable rate mortgages: Binomial tree versus analytical approximations

Article *in* Decisions in Economics and Finance · November 2012

DOI: 10.1007/s10203-011-0117-z



Werner Hürlimann
Swiss Mathematical Society

# VALUATION OF FIXED AND VARIABLE RATE MORTGAGES: BINOMIAL TREE VERSUS ANALYTICAL APPROXIMATIONS

Werner Hürlimann
FRSGlobal Switzerland AG
c/o. Regus Business Center
Seefeldstrasse 69, CH-8008 Zürich
E-mail : werner.huerlimann@frsglobal.com

## Abstract

The recombining binomial tree approach, which has been initiated by Cox, Ross and Rubinstein(1979) and extended to arbitrary diffusion models by Nelson and Ramaswamy(1990) and Hull and White(1990a), is applied to the simultaneous evaluation of price and Greeks for the amortized fixed and variable rate mortgage prepayment option. We consider the simplified binomial tree approximation to arbitrary diffusion processes by Costabile and Massabo(2010) and analyze its numerical applicability to the mortgage valuation problem for some Vasicek and CIR like interest rate models. For fixed rates and binomial trees with about thousand steps we obtain very good results. For the Vasicek model we also compare the closed-form analytical approximation of the callable fixed rate mortgage price by Xie(2009) with its binomial tree counterpart. With respect to the binomial tree values one observes a systematic underestimation (overestimation) of the callable mortgage price (prepayment option price) analytical approximation. This numerical discrepancy increases at longer maturities and becomes impractical for a valuable estimation of the prepayment option price.

## Keywords

Callable mortgage, prepayment option, Vasicek model, Hull-White model, CIR model, recombining binomial tree, analytical limiting formulas, analytical approximations

## JEL Classification

C61, C63, D81, G21

## 1. Introduction

The valuation of the callable mortgage with its prepayment (and default) options is a difficult but widely discussed problem. Two recent thesis entirely devoted to this subject are Goncharov(2003) and Sharp(2006) (see also Goncharov(2004/05) and Sharp et al.(2008/09)). Among the many divers approaches we focus in this study on the option based approach, which has been discussed earlier among others by Siegel(1984), Hall(1985), Kau et al.(1992/93), Kau and Keenan(1995), Dickinson and Heuson(1994), Stanton(1995), Kalotay et al.(2004).

Since the prepayment option is of American type, there does not seem to exist to our knowledge any simple and general evaluation method, which yields simultaneously both price and Greeks. Pricing of the prepayment option can be classified into four categories: analytical methods, recombining binomial and trinomial trees, Monte Carlo simulation methods and finite difference methods. With the single exception of Agarwal et al.(2008) no analytical closed-form solution to the mortgage pricing problem is known. Some promising recent attempts by Xie(2008/09), Xie et al.(2007a/07b/10), and Lo et al.(2009), have been made to obtain analytical approximations for both the optimal prepayment rate and the callable fixed rate mortgage price within the Vasicek and the Cox-Ingersoll-Ross (CIR) model frameworks. In Section 4 the analytical approximations by Xie et al.(2007a) and Xie(2009) for the Vasicek interest rate model will be considered and tested in Section 5.1 against the recombining binomial tree methodology, which is the approach adopted in the present study. Another quite popular technique is Monte Carlo simulation applied in conjunction with the regression algorithm by Longstaff and Schwartz(2001), which has been recently extended to the computation of Greeks in the Bermudan case by Belomestny et al.(2007). Finally, it is known that pricing financial instruments can be done by solving partial differential equations (e.g. Tavella and Randall(2000), Duffy(2006)). In this situation it is possible to use Green's function (e.g. Büttler and Waldvogel(1996)) and apply various more or less sophisticated finite difference numerical techniques. In particular, for the callable mortgage with prepayment and default options the basic partial differential equation is formula (1.17) in Sharp(2006).

The CRR approach, which has been extended to arbitrary diffusion models by Nelson and Ramaswamy(1990) and Hull and White(1990a), is applied to the simultaneous evaluation of price and Greeks for the fixed and variable rate mortgages. We consider the simplified recent binomial tree approximation to diffusion processes by Costabile et al.(2009) and Costabile and Massabo(2010) and analyze its applicability to the mortgage valuation problem for the Vasicek and CIR interest rate models. For fixed rates and binomial trees with about thousand steps very good results are obtained. The fact that the Vasicek and CIR models do not fit the initial term structure is not a disadvantage because these models have been extended to do so (e.g. Hull and White(1990b) and Hull(2003), Chap. 23.9, for the extended Vasicek or Hull-White model, Brigo and Mercurio(2001) for the CIR++ model, Chen and Scott(2003) for the multifactor CIR model) and are included in our general approach. The paper is organized as follows.

Section 2 introduces the pricing models for the amortized fixed and variable rate mortgage contracts. Section 3 recalls the construction by Costabile and Massabo(2010) of computationally simple binomial trees and shows how it applies to the evaluation of prices and Greeks for non-callable and callable mortgages as well as the associated prepayment option. Section 4 is devoted to the fixed rate mortgage under the Vasicek/CIR models. We display some exact and limiting analytical formulas and approximations including a correction to a formula by Xie(2009) for the Vasicek model. Section 5 provides numerical comparisons, examples and further discussion.

## 2. Pricing models for the default-free amortized fixed and variable rate mortgages

The pricing of amortized fixed and variable rate mortgages with given interest payment cycle is considered. In a variable rate mortgage the contract rate is adjusted periodically in order to reflect prevailing interest rates. Let us state some main features of our approach. We assume equal interest and amortization payment dates but distinguish between instantaneous payments in a continuous time framework and recurring discrete payments following a given payment cycle in a discrete time framework. The borrower has the option to prepay the mortgage at an arbitrary date prior maturity, the so-called *prepayment option*. For simplicity, no penalty is charged to the borrower at prepayment, but this assumption can be removed. Usually, the borrower has another option, the so-called *default option*, which consists in forfeiting the contract in exchange of the physical good underlying the mortgage. If the borrower exercises this option, then the lender does not receive any stream of payment anymore, but has the right to retain all previously done payments in addition to the underlying good. The prepayment and default options are alternative to each other in the sense that if one has been exercised, the contract expires and the other one cannot be exercised anymore. The borrower's default risk can be taken into account by charging an appropriate default insurance premium (e.g. Schwartz and Torous(1992)). We assume that it is included in the mortgage's servicing fee (e.g. Schwartz and Torous(1991), p.284). A simultaneous treatment of both options can be found among others in Sharp et al.(2008), Chen et al.(2009), and their references. In our approach, the latter requires the construction of a two-dimensional binomial tree and goes beyond the scope of the present study. For an extended analysis we recommend De Rossi and Vargiolu(2010), Section 5.

### 2.1. Amortized fixed rate mortgage with given interest payment cycle

Consider the *amortized fixed rate mortgage* (AFRM) contract on an underlying good (e.g. house or another physical good). Assume that at time $t=0$ the contract holder borrows an *initial capital* $P$ at the *nominal instantaneous rate* $\rho>0$ and pays it back over the time interval $[0,T]$ with $T$ the *maturity* of the contract. We distinguish between a *continuous amortization rate* $A$ per unit time and a recurring *discrete amortization payment* $A_P$ at the end of each *interest payment cycle* of length $IP$. To enable later on calculations in a *discrete time setting* we assume that the interval $[0,T]$ is divided into $N$ discrete time steps $[(i-1)h, ih], i=1,...,N$, each of length $h=T/N$. Set $nP=T/IP$ (the number of recurring payments), $nI=IP/h$ (the number of discrete steps in each interest payment cycle), and *assume* that $1/IP$ and $nI$ are positive integers. Let $\rho_N=e^{\rho h}-1$ be the interest rate per discrete time step and let $\rho_P=\rho_N\cdot nI$ be the interest rate per interest payment cycle. In the continuous time setting one has the basic equation

$$P=A\cdot\int_0^T e^{-\rho t}dt=A\cdot\frac{1-e^{-\rho T}}{\rho}. \tag{2.1}$$

It corresponds in the discrete time framework to the relationship

$$P = A_P \cdot \sum_{k=1}^{nP} (1+\rho_P)^{-k} = \frac{A_P}{\rho_P} \cdot \left(1 - (1+\rho_P)^{-nP}\right). \tag{2.2}$$

The case (2.1) of continuous amortization payments is approximated in practical calculations by the limiting case (2.2) with interest payment cycle $IP = h$ for which $nI = 1$. In this situation one has the relationship

$$A_P = A \cdot \frac{\rho_N}{\rho}. \tag{2.3}$$

The borrower has the option to prepay the mortgage at an arbitrary date $t < T$. In case of prepayment the *outstanding loan balance* equals

$$L_t = A \cdot \int_t^T e^{-\rho(u-t)} du = A \cdot \frac{1 - e^{-\rho(T-t)}}{\rho}, \quad t \in [0,T], \tag{2.4}$$

in continuous time, resp.

$$L_{k \cdot IP} = A_P \cdot \sum_{j=k+1}^{nP} (1+\rho_P)^{-(j-k)} = \frac{A_P}{\rho_P} \cdot \left(1 - (1+\rho_P)^{k-nP}\right), \quad k = 0,\dots,nP-1, \quad L_T = 0, \tag{2.5}$$

in discrete time, where it represents the outstanding loan balance at time $t = k \cdot IP$. If the prepayment option is exercised, the borrower must prepay the current outstanding loan balance plus any accrued interest since the last recurring payment. The charged amount is called *face value* and it is denoted and defined for each discrete time step by

$$FV_{k \cdot nI + m} = (1 + \rho_N \cdot m) \cdot L_{k \cdot IP}, \quad k = 0,\dots,nP-1, \quad m = 0,\dots,nI-1, \quad FV_N = 0. \tag{2.6}$$

Once the option is exercised the contract terminates. The prepayment option is therefore a contingent claim of American type. If the borrower exercises the prepayment option at time $t$ (resp. at time $ih$ in discrete time) the lender receives immediately the face value $FV_t = L_t$ (resp. $FV_i$ in discrete time) instead of the future stream of payments at the amortization rate $A$ per unit time (resp. $A_P$ at the end of each interest payment cycle). The market value at time $t$ (resp. at time $ih$ in discrete time) of these future cash-flows, also called *non-callable mortgage price*, equals

$$V_t^{nc} = A \cdot \int_t^T P(t,u) du, \quad t \in [0,T], \tag{2.7}$$

in continuous time, resp.

$$V_i^{nc} = A_P \cdot \sum_{k=[i/nI]+1}^{nP} P(ih, k \cdot IP), \quad i = 0,\dots,N-1, \quad V_N^{nc} = 0, \tag{2.8}$$

in discrete time. In both cases $P(t,s)$ denotes the price at time $t$ of a zero-coupon bond with maturity $s \geq t$. The lender is exposed to the risk of early exercise at time $t < T$ of an American option to exchange $V_t^{nc}$ for $FV_t$ (resp. $V_i^{nc}$ for $FV_i$ in discrete time). While the value of $FV_t$ is deterministic the value of $V_t^{nc}$ depends upon the evolution of the term structure of interest rates (TSIR) described by the zero-coupon bond structure $P(t,s), s \in [t,T]$. The optimal exercise of the prepayment option is triggered by market conditions like interest rates falling under a certain level, called *optimal prepayment rate*.

For banks or mortgage companies who hold a large pool of such contracts with different outstanding loan balances, different maturity dates, or different payment schedules, it is crucial to know the fair value of AFRM contracts with prepayment option. The determination of this fair value is not a trivial task because it depends upon the behavior of the borrower, which may act in a financial rational way or not. Once the market value and the sensitivities or Greeks of each contract are known, the construction of hedging and risk management strategies can begin.

In a continuous time and complete market framework with filtered probability space $(\Omega, F, Q)$ the market value at time $t < T$ of the mortgage with prepayment option, also called *callable mortgage price*, equals (e.g. De Rossi and Vargiolu(2010), equation (3))

$$\begin{aligned} V_t^c &= V_t^{nc} - ess \sup_{\tau \in [t,T]} E_Q\left[ \exp\left( -\int_t^{\tau} r_u du \right) \cdot \left( V_\tau^{nc} - FV_\tau \right) \middle| F_t \right] \\ &= V_t^{nc} - E_Q\left[ \exp\left( -\int_t^{\hat{\tau}} r_u du \right) \cdot \left( V_{\hat{\tau}}^{nc} - FV_{\hat{\tau}} \right) \middle| F_t \right], \quad t \in [0,T] \end{aligned}, \tag{2.9}$$

where the ess sup is taken over all $(F_i)$-stopping times. The *instantaneous interest rate* is assumed to follow a diffusion process of the type

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{2.10}$$

with $W_t$ the standard Wiener process, and $\hat{\tau}$ is the *optimal stopping time* under the rational refinancing assumption. In later concrete calculations we assume for simplicity that the short rate is mean-reverting with drift $\mu(t, r_t) = \alpha(\beta - r_t)$, and the instantaneous standard deviation is either constant $\sigma(t, r_t) = \sigma$ (model of Vasicek(1977)) or of square-root type $\sigma(t, r_t) = \sigma\sqrt{r_t}$ (model of Cox-Ingersoll-Ross(1985) or CIR model). The condition $2\alpha\beta > \sigma^2$ for the CIR model guarantees that the process never touches zero and implies a stationary gamma distribution. Similar calculations can be done for the Hull-White model with $\mu(t, r_t) = \alpha(\beta(t) - r_t)$ and $\sigma(t, r_t) = \sigma$ following the specification and calibration in Hull(2003), Chap. 23.

It is important to note that in a pool of mortgages one can observe different prepayment times, optimal stopping times and non-optimal ones, for different borrowers. The pricing of such mortgages is done using the so-called *prepayment function*. Let $\tau$ be the (optimal or not) *prepayment time* of a "typical" single borrower, which is a $(F_t)$-stopping time, that is a random variable with values in the time interval $[0,T]$. Let $H(\cdot|\theta)$ be its cumulative distribution function with respect to $Q$ conditional on the state variable $\theta$, which is defined by $H(t|\theta) = Q(\tau \le t|\theta)$. We assume that $\tau$ has a conditional density $h(t|\theta) = H'(t|\theta)$. Then the *prepayment function*, also called hazard function or risk function, which is similar to the default intensity in credit risk, is defined by

$$\pi(t|\theta) = h(t|\theta) \cdot [1 - H(t|\theta)]^{-1}. \tag{2.11}$$

This function describes the density of prepayment at time $t$ conditional on $\theta$ and the fact that the borrower has not yet prepaid. The price of the callable mortgage with prepayment function is (e.g. De Rossi and Vargiolu(2010), p.28)

$$V_t^{c,pr} = V_t^{nc} - E_Q\left[\int_t^T \pi(u|\theta)\exp\left(-\int_t^u (r_v + \pi(v|\theta)dv)\right)\left(V_u^{nc} - FV_u\right)du \middle| F_t\right], \quad t \in [0,T]. \tag{2.12}$$

Observe that (2.12) is similar to the pricing formula for a security under credit risk where the recovery value $V_\tau^{nc} - FV_\tau$ is paid at default (e.g. Schönbucher(2000)).

In various model specifications the prepayment function depends on the fact that at a given time it is optimal or not to prepay, that is on the function

$$\theta_t = \begin{cases} 1, & t = \hat{\tau} \\ 0, & else \end{cases} \tag{2.13}$$

For example, in the model by Stanton(1995) one has $\pi(t|\theta) = \lambda + \eta\theta_t$: an exogenous (non-optimal) prepayment has a constant hazard function $\lambda$, and if it is optimal to prepay the hazard function is augmented by the constant $\eta$. In particular, solving the American option type valuation problem (2.9) is a necessary step towards solving the more general problem (2.12). In the following we focus solely on the equation (2.9).

We apply weak convergence of computationally simple trees to determine directly (2.9) as in De Rossi and Vargiolu(2010). Besides the Euler scheme for the stochastic differential equation the use of *binomial trees* is a simple and common calculation scheme in finance. In the discrete time setting the problem (2.9) consists to evaluate

$$ess \sup_{\tau \in [i,N]} E_Q\left[\exp\left(-\tfrac{T}{N}\sum_{k=i}^{\tau} r_k\right) \cdot \left(V_\tau^{nc} - FV_\tau\right) \middle| F_i\right], \quad i = 0,\ldots,N-1, \tag{2.14}$$

where the ess sup is taken over all the $(F_i)$-stopping times $\tau$ taking integer values between $i$ and $N$, $r_k$ is the short rate of type (2.10) taken at time $k \cdot h$, and $V_\tau^{nc} - FV_\tau$ is determined by (2.6) and (2.8) with the zero-coupon bond values

$$P(ih, k \cdot IP) = E_Q\left[\exp\left(-\tfrac{T}{N}\sum_{j=i}^{k-1} r_j\right) \middle| F_i\right], \quad i = 0,\ldots,N, \quad k = \left[\tfrac{i}{nI}\right],\ldots,nP. \tag{2.15}$$

If the short rates $(r_j)$ build now a Markov chain, then the quantities (2.15) are deterministic functions of $i$ and $r_i$. The conditional expectation in (2.14) is a function which only depends on $i$, $r_i$ and $V_i^{nc} - FV_i$. In this situation (2.14) can be evaluated by backward recursion using the Snell envelope as follows (e.g. De Rossi and Vargiolu(2010), formula (8))

$$\begin{aligned} &U_N(r) := V_N^{nc}(r) - FV_N(r) \equiv 0, \\ &U_i(r) := \max\left\{V_i^{nc}(r) - FV_i(r), E_Q\left[\exp\left(-\tfrac{T}{N} r_{i+1}\right)\cdot U_{i+1}(r_{i+1}) \middle| r_i = r\right]\right\}, \\ &i = N-1, N-2, \ldots, 2, 1, 0, \end{aligned} \tag{2.16}$$

where an optimal stopping time is determined by

$$\hat{\tau} := \inf\left\{i \le N \middle| U_i(r_i) = V_i^{nc}(r_i) - FV_i(r_i)\right\}. \tag{2.17}$$

Given the initial short rate $r_0$ the computational cost for the evaluation of the functions $(U_i)$ depends on the chosen model for the discrete time evolution of the short rate. Applying binomial trees the state space for the short rate is a finite set, but its cardinality depends on the type of tree. If the tree is not recombining the state space for $r_i$ consists of up to $2^i$ points. If it is recombining the state space grows at most linearly with $i$. It appears thus most efficient to use a recombining tree dynamics for the short rate. We follow in Section 3 the approaches by Nelson and Ramaswamy(1990) and Hull and White(1990a) as simplified in Costabile et al.(2009) and Costabile and Massabo(2010).

## 2.2. Amortized variable rate mortgage with given interest payment cycle

Consider now the amortized variable rate mortgage (AVRM) contract with continuous and discrete amortization payments at equal dates of interest and amortization payments. In contrast to the fixed nominal interest rate $\rho > 0$ of an AFRM contract, which by given initial capital $P$ determines the fixed continuous amortization rate $A$ in continuous time, resp. the fixed recurring discrete amortization payment $A_P$ in discrete time, our AVRM contract is based on a variable deterministic nominal interest rate $\rho(s) > 0$ at time $s \in [0,T]$. The variable rate leads to a variable continuous amortization rate $A(s)$ per unit time, resp. to variable recurring discrete amortization payment $A_P(s)$ at the end of each payment cycle of length $IP$. We suppose yearly adjustments of the rate $\rho(s)$ at the times $s = 0,\ldots,T-1$, that are settled at the initial time of contract agreement, such that $A(s)$, resp. $A_P(s)$, are fixed over the time periods $(s, s+1]$, $s = 0,\ldots,T-1$. However, the variability in payments is partly offset due to the presence of a *lifetime cap* $\ell$ on the contract rate as well as a *periodic cap and floor* $y$ that limits the possible change in contract rate at each adjustment date. It is usual to assume that the contract rate changes according to an index, where the precise specification may vary. Inspired by Sharp(2006), Section 2.3.1, we use a modified (forward) index, which depends on the initial observed TSIR and a margin (denoted *margin*). Another example is found in Stanton and Wallace(1995), which use an index that lags behind shifts in the term structure. Moreover, there is often an initial teaser rate (denoted *teaser*) such that the initial contract rate is artificially set below the rate that the contract's rules would otherwise offer. A detailed specification follows.

In practice, a deterministic set of variable one-year contract rates can be obtained from the initial bond price structure as follows. The (forward) *index* at time $s = 0,...,T-1$, denoted by $IY^F(s)$, is the implied forward mortgage-equivalent rate of a one year default-free pure discount bond with maturity $T > s$. It is defined by

$$IY^F(s) = \frac{1}{IP} \cdot \left(e^{R(0;s,T)\cdot IP} - 1\right), \quad R(0;s,T) = -\frac{\ln P(0,T) - \ln P(0,s)}{T-s}, \tag{2.18}$$

where $R(0;s,T)$, $s = 0,...,T-1$, denotes the continuously compounded forward rate for the time period $[s,T]$ contracted at the initial date of agreement (e.g. Björk(1998), Definition 15.2, p.230). The variable one-year contract rate, denoted by $\rho_D(s)$, is defined recursively by

$$\begin{aligned} &\rho_D(0) = IY^F(0) + margin - teaser, \\ &\rho_D(s) = \max\left\{ \begin{matrix} \min\left(IY^F(s) + margin, \rho_D(s-1) + y, \rho_D(0) + \ell\right), \\ \rho_D(s-1) - y \end{matrix} \right\}, \ s = 1,...,T-1. \end{aligned} \tag{2.19}$$

At each adjustment date, the new contract rate is equal to the current value of the interest rate dependent index plus the margin, as long as this value does not increase beyond the initial level by more than the lifetime cap, or deviate from the previous contract rate by more than the periodic cap and floor. Similarly to the treatment in Section 2.1, let $\rho_P(s) = \rho_D(s) \cdot IP$ be the interest rate per interest payment cycle over the time period $(s, s+1]$, and let $\rho_N(s) = \rho_D(s) \cdot h$ be the interest rate per discrete time step. A corresponding approximate continuous contract rate is $\rho(s) = \ln\{1 + \rho_N(s)\}/h$. Once the contract rate is known at the beginning of the time period $(s, s+1]$, the recurring discrete amortization payment $A_P(s)$ at the end of each payment cycle over this time period is determined by the current outstanding loan balance $L_s$ and the current contract rate $\rho_P(s)$. This amortization payment is calculated under the assumption that the mortgage at time $s$ is a fixed rate mortgage contract that fully amortizes the current outstanding balance over the remaining life of the loan, which implies that

$$A_P(s) = L_s \cdot \frac{\rho_P(s)}{1 - \left(1 + \rho_P(s)\right)^{-\left(\frac{T-s}{IP}\right)}}, \quad s = 0,...,T-1. \tag{2.20}$$

The outstanding loan balance at time $t = s + k \cdot IP$ after the $k$-th amortization payment in the time period $(s, s+1]$ is determined by

$$L_{s+k\cdot IP} = L_s \cdot \frac{1-(1+\rho_P(s))^{k-(\frac{T-s}{IP})}}{1-(1+\rho_P(s))^{-(\frac{T-s}{IP})}}, \quad s=0,...,T-1, \quad k=0,...,\frac{1}{IP}, \tag{2.21}$$

where at initial time one has $L_0 = P$, the initial capital. If the prepayment option is exercised, the face value charged to the borrower at each discrete time step is given by (generalized equation (2.6))

$$\begin{aligned} &FV_{\frac{s}{h}+k\cdot nI+m} = (1+\rho_N(s)\cdot m)\cdot L_{s+k\cdot IP}, \\ &s=0,...,T-1, \quad k=0,...,\frac{1}{IP}-1, \quad m=0,...,nI-1, \quad FV_N=0. \end{aligned} \tag{2.22}$$

Similarly to (2.8), the non-callable mortgage price at time $t = s + k\cdot IP$ is determined by

$$\begin{aligned} &V^{nc}_{\frac{s}{h}+k\cdot nI} = A_P(s)\cdot \sum_{\ell=k+1}^{1/IP-1} P(s+k\cdot IP, s+\ell\cdot IP) + \sum_{\tau=s+1}^{T-1} A_P(\tau)\cdot \sum_{\ell=0}^{1/IP-1} P(s+k\cdot IP, \tau+\ell\cdot IP), \\ &s=0,...,T-1, \quad k=0,...,1/IP-1, \end{aligned} \tag{2.23}$$

with the convention that an empty sum is zero. At time $T$ one has $V_N^{nc}=0$. To calculate the callable mortgage price defined in (2.9) we use again the backward recursion (2.16). A special case of the above structure has been considered in De Rossi and Vargiolu(2010), Section 3.5.

## 3. <u>Prices and Greeks from computationally simple binomial trees</u>

Recall the modification by Costabile and Massabo(2010) of the Cox and Rubinstein(1985) binomial approach to obtain a direct discrete scheme of the original heteroscedastic process (2.10) by means of a binomial tree with a number of nodes that grows linearly with the number of steps. Let $r_0$ be the initial value of the discrete time binomial approximation of the diffusion process. After the first time step, the process may jump up to $r_1^u = r_0 + \sigma(r_0)\sqrt{h}$ or down to $r_1^d = r_0 - \sigma(r_0)\sqrt{h}$. At the next time step, the process is forced to take one of three values:

$r_2^{uu} = r_1^u + \sigma(r_1^u)\sqrt{h}$ : two consecutive upward jumps
$r_2^{dd} = r_1^d - \sigma(r_1^d)\sqrt{h}$ : two consecutive downward jumps
$r_2^{ud} = r_2^{du} = r_0$ : upward (downward) jump followed by a downward (upward) jump

To describe the evolution of the discrete process on the whole binomial tree, one uses the following state space notation:

$r(i,j)$ : value of the *binomial process* at the node $(i,j)$ reached after $i$ *time steps* with $j$ *upward steps* and $i-j$ *downward steps*, $i=0,...,N,\ j=0,...,i$

In this notation one has at initial time $r(0,0)=r_0$, after one time step

$$r(1,1)=r(0,0)+\sigma(r(0,0))\sqrt{h}\ ,\quad r(1,0)=r(0,0)-\sigma(r(0,0))\sqrt{h}\ ,$$

and after two time steps

$$r(2,2)=r(1,1)+\sigma(r(1,1))\sqrt{h}\ ,\quad r(2,1)=r(0,0),\quad r(2,0)=r(1,0)-\sigma(r(1,0))\sqrt{h}\ .$$

This discrete scheme continues this way until the last time step $N$ is reached by setting successively for the nodes located on the upper edge

$$r(i,i)=r(i-1,i-1)+\sigma(r(i-1,i-1))\sqrt{h}\ , i=1,\ldots,N, \tag{3.1}$$

for the nodes located on the lower edge

$$r(i,0)=\max\left\{r(i-1,0)-\sigma(r(i-1,0))\sqrt{h},0\right\}, i=1,\ldots,N, \tag{3.2}$$

and for the nodes located on the internal nodes

$$r(i,j)=\begin{cases} r(0,0)=r_0, & i-j=j, \\ r(2j-i,2j-i), & i-j<j, \\ r(i-2j,0), & i-j>j. \end{cases} \tag{3.3}$$

For example, the binomial tree of the Vasicek short rate diffusion process is obtained setting $r(0,0)=r_0$ and applying the following recursive scheme for $i=1,\ldots,N,\ j=1,\ldots,i$:

$$\begin{aligned} &r(i,0)=\max\left\{r(i-1,0)-\sigma\sqrt{h},0\right\},\quad r(i,i)=r(i-1,i-1)+\sigma\sqrt{h}, \\ &r(i,j)=r(i-2,j-1),\ if\ \ j<i. \end{aligned} \tag{3.3'}$$

The internal nodes are defined by generating horizontal layers of nodes, each one beginning at a node located on an upper or lower edge. Since negative values of the nominal interest rates have no economic significance, (3.2) shows that the approximating tree is truncated at the lower zero boundary. Later on this is taken into account using the following *truncation index*

$$index(0)=0,\quad index(i)=\sum_{j=0}^{i-1}1\{r(i,j)-r(i,j+1)=0\},\quad i=1,\ldots,N, \tag{3.4}$$

where $1\{\ldots\}$ is the indicator function.

It remains to define the *transition probabilities* associated with each node. A natural choice is to define these so that the local mean of the discrete process matches the drift of the limiting diffusion. This procedure yields the following probability for an upward jump at node $(i,j)$

$$p(i,j)=\frac{\mu(r(i,j))h+r(i,j)-r(i+1,j)}{r(i+1,j+1)-r(i+1,j)}. \tag{3.5}$$

Unfortunately, this simple device does not define in general a legitimate probability, because the value (3.4) may fall outside the interval $[0,1]$. To overcome this difficulty multiple upward and downward jumps must be considered. A *multiple upward jump* at node $(i,j)$ is defined by

$J^u(i,j)$ : the smallest positive integer $j^* \in [1,1+i]$ such that $r(i+1,j^*) \geq \mu(r(i,j))h + r(i,j)$

A *multiple downward jump* at node $(i,j)$ is defined by $J^d(i,j)=J^u(i,j)-1$. With this the *probability for an upward jump* at node $(i,j)$ equals in general

$$\begin{aligned}
&p(i,j)=\max\left(\frac{\mu(r(i,j))h+r(i,j)-r(i+1,J^d(i,j))}{r(i+1,J^u(i,j))-r(i+1,J^d(i,j))},0\right), \quad index(i)\leq j\leq i,\\
&p(i,j)=0, \quad j<index(i).
\end{aligned} \tag{3.6}$$

Clearly, the *probability for a downward jump* at node $(i,j)$ is $1-p(i,j)$.

As an application of this simple binomial interest rate tree model let us restate now the recursive algorithm (2.16)-(2.17) for the approximate numerical evaluation of the prepayment option associated to the AFRM/AVRM contracts and the corresponding optimal decisions for prepayment. First, one defines a *matrix table* $MV^{nc}(i,j)$ *of non-callable AFRM/AVRM market values* over the binomial interest rate tree by considering the present values at the nodes $(i,j)$ of the current and future cash-flows. This table is generated as follows. Let $CF_i,\ i=0,...,N$ be the *deterministic cash-flows* of the AFRM contract (AVRM contract) at each time step defined by $CF_i = A_P$ if $i=k\cdot nI,\ k=1,...,nP$, ( $CF_i = A_P(s)$ if $i=\frac{s}{h}+k\cdot nI,\ s=0,...,T-1,\ k=1,...,\frac{1}{IP}$ ), and $CF_i=0$ otherwise. For $j=0,...,N$ one sets $MV^{nc}(N,j)=CF_N$ in case $index(N-1)\leq j\leq N$ and $MV^{nc}(N,j)=0$ otherwise. Then one has recursively

$$\begin{aligned}
&MV^{nc}(i,j)=CF_i+e^{-r(i,j)h}\cdot\left[p(i,j)\cdot MV^{nc}(i+1,J^u(i,j))+(1-p(i,j))\cdot MV^{nc}(i+1,J^d(i,j))\right],\\
&i=N-1,...,1, \quad index(i-1)\leq j\leq i, \quad MV^{nc}(i,j)=0, \quad j<index(i-1),\\
&MV^{nc}(0,0)=CF_0+e^{-r_0h}\cdot\left[p(0,0)\cdot MV^{nc}(1,1)+(1-p(0,0))\cdot MV^{nc}(1,0)\right]
\end{aligned} \tag{3.7}$$

The market value $MV^{nc}(0,0)$ at initial time is an approximation of $V_0^{nc}$. Next, one generates a *matrix table* $IV(i,j)=\max\{MV^{nc}(i,j)-FV_i,0\}$ *of intrinsic values of the prepayment option*, where the face value $FV_i$ is defined in (2.6) for the AFRM contract, respectively (2.22) for the

AVRM contract. From this one obtains a *matrix table* $CV(i,j)$ *of continuation values of the prepayment option* defined by

$$\begin{aligned}
&CV(N,j)=0,\ j=0,\dots,N,\\
&CV(i,j)=\max\left\{\begin{matrix} IV(i,j), e^{-r(i,j)h}\cdot p(i,j)\cdot CV\left(i+1,J^{u}(i,j)\right)\\ +e^{-r(i,j)h}\cdot(1-p(i,j))\cdot CV\left(i+1,J^{d}(i,j)\right)\end{matrix}\right\},\\
&i=N-1,\dots,1,\quad index(i-1)\le j\le i,\quad CV(i,j)=0,\quad j<index(i-1),\\
&CV(0,0)=e^{-r_0 h}\cdot\{p(0,0)\cdot CV(1,1)+(1-p(0,0))\cdot CV(1,0)\}
\end{aligned}\tag{3.8}$$

It is known that the considered binomial process weakly converges to the diffusion process (2.10) (for a proof consult the papers by Costabile et al.(2009) and Costabile and Massabo(2010)). Therefore, the above calculations yield the following approximations at initial time $t=0$: the *non-callable mortgage price* $V_0^{nc}\cong MV^{nc}(0,0)$, the *prepayment option value* $V_0^{po}\cong CV(0,0)$ and the *callable mortgage price* $V_0^{c}\cong MV^{nc}(0,0)-CV(0,0)$. Binomial interest rate tree approximations of the *optimal stopping times* and the *optimal prepayment rate* are also obtained from the *matrix table* of indicators $\tau(i,j)=1\{CV(i,j)=IV(i,j)>0\}$ *of optimal decisions of prepayment*. According to the binomial tree translation of (2.17) the optimal stopping time is attained at the first index $i\le N$ for which there exists $j\le i$ satisfying $\tau(i,j)=1$. Since $r(i,j)$ increases in the second argument by fixed first argument, the optimal prepayment rate is attained at the node with largest $j\le i$ satisfying $\tau(i,j)=1$.

Binomial interest rate trees have the advantage to provide additionally simple approximations for the price sensitivities or Greeks of financial instruments including options of American type. The most important Greeks are the *delta* $\Delta$ (sensitivity with respect to a change in the current market rate), the *gamma* $\Delta$ (rate of change of delta) and *theta* $\Theta$ (sensitivity to passage of time or time value). At initial time $t=0$ one has the following well-known formulas

$$\Delta^{po}\cong\frac{CV(1,1)-CV(1,0)}{r(1,1)-r(1,0)},\quad \Delta^{nc}\cong\frac{MV^{nc}(1,1)-MV^{nc}(1,0)}{r(1,1)-r(1,0)},\quad \Delta^{c}\cong\Delta^{nc}-\Delta^{po}\tag{3.9}$$

$$\Gamma^{po}\cong\frac{\frac{CV(2,2)-CV(2,1)}{r(2,2)-r(2,1)}-\frac{CV(2,1)-CV(2,0)}{r(2,1)-r(2,0)}}{0.5(r(2,2)-r(2,0))},\ \Gamma^{nc}\cong\frac{\frac{MV^{nc}(2,2)-MV^{nc}(2,1)}{r(2,2)-r(2,1)}-\frac{MV^{nc}(2,1)-MV^{nc}(2,0)}{r(2,1)-r(2,0)}}{0.5(r(2,2)-r(2,0))},\ \Gamma^{c}\cong\Gamma^{nc}-\Gamma^{po}\tag{3.10}$$

$$\Theta^{po}\cong\frac{CV(2,1)-CV(0,0)}{2\cdot h},\quad \Theta^{nc}\cong\frac{MV^{nc}(2,1)-MV^{nc}(0,0)}{2\cdot h},\quad \Theta^{c}\cong\Theta^{nc}-\Theta^{po}\tag{3.11}$$

Improved methods to increase accuracy or accelerate convergence in binomial trees are found in Wallner and Wystup(2004) and De Rozario(2004) among others. Often, one extends the binomial tree on the left and use adjusted formulas for theta (e.g. Hull(1993), Chung and Shackelton(2002)).

## 4. Some analytical approximations for the Vasicek & CIR models

While the binomial tree calculations of Section 3 apply to all diffusion processes of the type (2.10) their accuracy in dependence of the crucial number of steps $N$ can be tested through comparison with the best known numerical or analytical approximations available. For the Vasicek and CIR models there are a lot of recent studies, which provide more or less reliable results for the AFRM contract with continuous amortization payments (e.g. Xie(2008/09), Xie et al.(2007a/07b/10), Lo et al.(2009)).

A first check of the accuracy of the binomial tree approximation can be done through comparison with the exact analytical bond price formulas as well as other analytical approximations for the price and Greeks of the corresponding non-callable mortgage. To begin with the zero-coupon bond price $P(0,T)$ note that a binomial tree approximation following Section 3 yields the recursive scheme

$$\begin{aligned}
&B(N,j)=1, \quad index(N-1)\le j\le N, \quad B(N,j)=0, \quad j<index(N-1), \\
&B(i,j)=e^{-r(i,j)h}\cdot\left[p(i,j)\cdot B\left(i+1,J^{u}(i,j)\right)+\left(1-p(i,j)\right)\cdot B\left(i+1,J^{d}(i,j)\right)\right], \\
&i=N-1,\ldots,1, \quad index(i-1)\le j\le i, \quad B(i,j)=0, \quad j<index(i-1).
\end{aligned} \tag{4.1}$$

At initial time $t=0$ one has the approximation

$$P(0,T)\cong B(0,0)=e^{-r_0 h}\cdot\left[p(0,0)\cdot B(1,1)+\left(1-p(0,0)\right)\cdot B(1,0)\right]. \tag{4.2}$$

On the other side, the exact bond price formulas are well-known. For the Vasicek process $dr_t=\alpha(\beta-r_t)dt+\sigma dW_t$ one has

$$P(0,T)=e^{A(T)-B(T)\cdot r_0}, \quad B(u)=\frac{1-e^{-\alpha u}}{\alpha}, \quad A(u)=\left(\beta-\frac{1}{2}\left(\frac{\sigma}{\alpha}\right)^2\right)\left(B(u)-u\right)-\frac{1}{4}\frac{[\sigma B(u)]^2}{\alpha}. \tag{4.3}$$

while for the CIR process $dr_t=\alpha(\beta-r_t)dt+\sigma\sqrt{r_t}\,dW_t$ one has

$$\begin{aligned}
&P(0,T)=e^{A(T)-B(T)\cdot r_0}, \quad B(u)=\frac{2\left(e^{\gamma u}-1\right)}{(\gamma+\alpha)\left(e^{\gamma u}-1\right)+2\gamma}, \\
&A(u)=\frac{2\alpha\beta}{\sigma^2}\ln\left\{\frac{2\gamma e^{\frac{1}{2}(\gamma+\alpha)u}}{(\gamma+\alpha)\left(e^{\gamma u}-1\right)+2\gamma}\right\}, \quad \gamma=\sqrt{\alpha^2+2\sigma^2}.
\end{aligned} \tag{4.4}$$

Costabile and Massabo(2010) have tested their binomial tree against formula (4.4) for the CIR process and have shown almost accuracy for $N=1000$ steps. It is also possible to test analytical approximations for the non-callable mortgage price against the binomial tree value $V_0^{nc}\cong V^{nc}(0,0)$ obtained from (3.7). For this, it suffices to remark that this price can be expressed as limiting discrete analytical approximation to the integral (2.7) as

$$V_0^{nc} = A \cdot \int_0^T P(0,u)du = A \cdot \lim_{h_M \to 0} h_M \cdot \sum_{k=1}^{M} e^{A(kh_M) - B(kh_M) \cdot r_0}, \quad h_M = \frac{T}{M}. \tag{4.5}$$

Similar expressions can be obtained for the Greeks of the non-callable mortgage. One observes that (4.8) is in virtue of (4.3) and (4.4) an exact analytical formula.

$$\Delta^{nc} = \left.\frac{\partial V_0^{nc}}{\partial r}\right|_{r=r_0} = -A \cdot \lim_{h_M \to 0} h_M \cdot \sum_{k=1}^{M} B(kh_M) \cdot e^{A(kh_M) - B(kh_M) \cdot r_0}. \tag{4.6}$$

$$\Gamma^{nc} = \left.\frac{\partial^2 V_0^{nc}}{\partial r^2}\right|_{r=r_0} = A \cdot \lim_{h_M \to 0} h_M \cdot \sum_{k=1}^{M} B(kh_M)^2 \cdot e^{A(kh_M) - B(kh_M) \cdot r_0}. \tag{4.7}$$

$$\Theta^{nc} = \left.\frac{\partial V_t^{nc}}{\partial t}\right|_{t=0} = A \cdot \frac{\partial}{\partial t}\int_t^T P(t,u)du\bigg|_{t=0} = A \cdot \frac{\partial}{\partial t}\int_0^{T-t} P(0,\tau)d\tau\bigg|_{t=0} = -A \cdot P(0,T) \tag{4.8}$$

All these analytical exact formulas and limiting approximations are tested below in Section 5.1.

In the above cited references one also finds various numerical and analytical results for the optimal prepayment rate and the callable mortgage price of the AFRM contract with continuous amortization payments. We restrict ourselves to Xie et al.(2007a) and Xie(2009), who propose analytical approximations for the Vasicek interest rate model.

Consider first the optimal prepayment rate at initial time $t = 0$, which as function of the maturity $T$ is denoted by $R(T)$. Bian et al.(2005) study the asymptotic behavior of the optimal prepayment rate near expiry or equivalently as $T \to 0$ and obtain the result (see also Xie et al.(2007a), Theorem 3)

$$R(T) \sim \rho - \sigma\kappa\sqrt{2T} \quad as \quad T \to 0, \tag{4.9}$$

where the constant $\kappa \cong 0.33436$ is the unique root of the integral equation

$$\sqrt{\pi} = \int_0^{\kappa} \frac{e^{-z^2}\left(\kappa^2 - z^2\right)^4\left(18\kappa^2 + 2z^2\right)}{\left(\kappa^2 + z^2\right)^5} dz. \tag{4.10}$$

Xie et al.(2007a) study the asymptotic behavior for an infinite maturity and obtain the analytical result (see Theorem 4 and formula (7.18))

$$R(T) \sim R^* + \rho^* e^{-\rho T} \quad as \quad T \to \infty, \tag{4.11}$$

$$R^* = \beta - \left(\frac{\sigma}{\alpha}\right)^2 + \frac{\sigma}{\sqrt{\alpha}} x^*, \quad \rho^* = \frac{\sigma^2}{2\alpha(\rho - R^*)} - \frac{\sigma\sqrt{\alpha}}{2(\rho - R^*)} \cdot \frac{H_x\left(\mu + \frac{\rho}{\alpha}, x^*\right)}{H\left(\mu + \frac{\rho}{\alpha}, x^*\right)} \tag{4.12}$$

with $\mu = \frac{1}{\alpha}\left(\frac{1}{2}\left(\frac{\sigma}{\alpha}\right)^2 - \beta\right)$, $H(\nu, x)$ the Hermite function, and $x^*$ implicit solution of

$$\beta-\left(\frac{\sigma}{\alpha}\right)^2+\frac{\sigma}{\sqrt{\alpha}}\frac{\int_{x^*}^{\infty}yH(\mu,y)e^{-y^2+\xi y}dy}{\int_{x^*}^{\infty}H(\mu,y)e^{-y^2+\xi y}dy}=\rho,\quad \xi=\frac{\sigma}{\alpha\sqrt{\alpha}}. \tag{4.13}$$

Combining (4.9) and (4.11) two simple global approximations are derived in Xie et al.(2007a), Section 8. For an approximation of the form

$$R_I(T)=\rho-\sigma\kappa\sqrt{2\frac{1-e^{-bT}}{b}}, \tag{4.14}$$

which satisfies (4.9) and the behavior $R(T)\sim R^*\quad as\quad T\to\infty,$ one finds (Xie et al.(2007a), formula (8.1))

$$R_I(T)=\rho-\sigma(\rho-R^*)\sqrt{1-\exp\left\{2\left(\frac{\kappa\sigma}{\rho-R^*}\right)^2T\right\}}. \tag{4.15}$$

On the other side, using the more detailed information (4.11)-(4.13), one shows the enhanced approximation

$$R_{II}(T)=\rho-\sigma\kappa\sqrt{\frac{1-e^{-2\rho T}}{\rho}}+\rho^*\left(e^{-\rho T}-e^{-2\rho T}\right)+\left(R^*-\rho+\frac{\kappa\sigma}{\sqrt{\rho}}\right)\left(1-e^{-2\rho T}\right). \tag{4.16}$$

For a typical parameter set Xie et al.(2007a) find a relative error of the small order of magnitude

$$\frac{\max_{t<T}\{R(t)-R_{II}(t)\}}{\rho-R^*}\approx 0.4\%. \tag{4.17}$$

From the contract holder's point of view it is even more important to know the market value $V^c(x,T)$ or price of the callable AFRM contract given its maturity $T$ and the current market return $x=r_0$ . Assuming $A=1$ in (1.1) and making the change of variable $V^c(x,T)=V(y,T),\ y=x-R(T)$ , Xie(2009), formulas (21)-(22), derives the two asymptotic expansions

$$V(y,T)\sim a+by^2+cy^3,\ y\to 0,\quad V(y,T)\sim \tfrac{1}{y},\ y\to\infty, \tag{4.18}$$

with

$$a=\tfrac{1}{\rho}\left(1-e^{-\rho T}\right),\quad b=-\tfrac{1}{\sigma^2}\left(1-e^{-\rho T}\right)\left(1-\tfrac{R(T)}{\rho}\right),\quad c=\tfrac{1}{3\sigma^2}\left(a-\alpha(\beta-R(T))b\right). \tag{4.19}$$

From this Xie(2009), Section 3, derives two analytical approximations. Restrict the attention to the second more accurate one, which uses the full asymptotic information contained in (4.18). For this consider an approximation of the type

$$V(y,T) = (P_1 + \lambda_1 y) ERFCX(Q_1 y^2) + (P_2 + \lambda_2 y) ERFCX(Q_2 y^2), \quad y \geq 0, \tag{4.20}$$

with $ERFCX(z) = e^{z^2} ERFC(z)$, $ERFC(z) = 2\overline{\Phi}(\sqrt{2}z)$, $\overline{\Phi}(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-z^2} dz$, the scaled complementary error function. To approximate (4.20) the asymptotic expansions (e.g. Jeffrey and Zwillinger(2000), 890-892)

$$\begin{aligned} &ERFCX(z) \sim \frac{1}{\sqrt{\pi} z}\left(1 - \frac{1}{2z^2} + \frac{1 \cdot 3}{(2z^2)^2 \mp \ldots}\right) \sim \frac{1}{\sqrt{\pi} z}, \; z \to \infty, \\ &ERFCX(z) \sim 1 - \frac{2}{\sqrt{\pi}} z, \; z \to 0, \end{aligned} \tag{4.21}$$

are introduced in (4.20), and this is compared with (4.18) to get the system of equations

$$\begin{aligned} &\frac{\lambda_1}{Q_1} + \frac{\lambda_2}{Q_2} = \sqrt{\pi}, \quad P_1 + P_2 = a, \quad \lambda_1 + \lambda_2 = 0, \\ &-\frac{2}{\sqrt{\pi}}(P_1 Q_1 + P_2 Q_2) = b, \quad -\frac{2}{\sqrt{\pi}}(\lambda_1 Q_1 + \lambda_2 Q_2) = c, \quad Q_1, Q_2 > 0. \end{aligned} \tag{4.22}$$

To solve this set $P_1 = x_1 a, \, P_2 = x_2 a, \quad x_1 + x_2 = 1.$ Inserting into the 4$^{\text{th}}$ equation one gets

$$x_1 Q_1 + x_2 Q_2 = -\frac{\sqrt{\pi}}{2} \frac{b}{a}. \tag{4.23}$$

Elimination of $\lambda_1, \lambda_2$ in the 1$^{\text{st}}$, 3$^{\text{rd}}$ and 5$^{\text{th}}$ equations yields further $2Q_1 Q_2 = c$. This allows for elimination of $Q_2$ in (4.23) and yields the quadratic equation for $Q_1$

$$2x_1 Q_1^2 + \sqrt{\pi} \frac{b}{a} Q_1 + x_2 C = 0, \tag{4.24}$$

which has a real solution if and only if its determinant is non-negative, that is

$$\Delta(x_1) = \pi\left(\tfrac{b}{a}\right)^2 + 8x_1^2 c - 8x_1 c \geq 0. \tag{4.25}$$

The limiting case $\Delta(x_1) = 0$ yields the solution

$$x_1 = \tfrac{1}{2}\left(1 + \sqrt{1 - \tfrac{\pi}{2C}\left(\tfrac{b}{a}\right)^2}\right), \quad c > \tfrac{\pi}{2}\left(\tfrac{b}{a}\right)^2, \quad x_2 = 1 - x_1. \tag{4.26}$$

If the condition $c > \frac{\pi}{2}\left(\frac{b}{a}\right)^2$ is not satisfied, then choose simply $x_1 = x_2 = \frac{1}{2}$, which implies that $\Delta\left(\frac{1}{2}\right) = \pi\left(\frac{b}{a}\right)^2 - 2c \geq 0$. In this situation (4.24) has a real solution. Summarizing the analysis, the coefficients in (4.20) can be specified explicitly and uniquely as follows:

Case 1: $c \le \frac{\pi}{2}\left(\frac{b}{a}\right)^2$

$$P_1 = P_2 = \tfrac{1}{2}a, \quad Q_1 = \tfrac{1}{2}\left(-\sqrt{\pi}\,\tfrac{b}{a} - \sqrt{\pi\left(\tfrac{b}{a}\right)^2 - 2c}\right), \\ Q_2 = \tfrac{1}{2Q_1}c, \quad \lambda_1 = \tfrac{\sqrt{\pi}}{2}\tfrac{1}{Q_2 - Q_1}c, \quad \lambda_2 = -\lambda_1 \tag{4.27}$$

Case 2: $c > \frac{\pi}{2}\left(\frac{b}{a}\right)^2$

$$P_1 = x_1 a, \quad P_2 = x_2 a, \quad x_1 = \tfrac{1}{2}\left(1 + \sqrt{1 - \tfrac{\pi}{2C}\left(\tfrac{b}{a}\right)^2}\right), \quad x_2 = 1 - x_1, \\ Q_1 = -\tfrac{1}{4x_1}\sqrt{\pi}\,\tfrac{b}{a}, \quad Q_2 = \tfrac{1}{2Q_1}c, \quad \lambda_1 = \tfrac{\sqrt{\pi}}{2}\tfrac{1}{Q_2 - Q_1}c, \quad \lambda_2 = -\lambda_1 \tag{4.28}$$

We note that the simplifying choice $x_1 = x_2 = \frac{1}{2}$ is proposed in Xie(2009). However, this author does not state (4.27) and misses to mention that this simplifying assumption does not lead to a solution in case $c > \frac{\pi}{2}\left(\frac{b}{a}\right)^2$. With (4.20) the approximation of the callable mortgage price reads

$$V_I^c(x,T) = \begin{cases} (P_1 + \lambda_1 y)ERFCX\left(Q_1 y^2\right) + (P_2 + \lambda_2 y)ERFCX\left(Q_2 y^2\right), & y = x - R(T) > 0, \\ a, \quad y = x - R(T) \le 0. \end{cases} \tag{4.29}$$

## 5. Numerical examples

We illustrate numerically the findings of the preceding Sections for the Vasicek and CIR models of the TSIR with the following parameters:

| parameters \ TSIR model | Vasicek | CIR |
|---|---|---|
| speed of reversion α | 0.15 | 0.15 |
| long term mean level β | 5% | 5% |
| instantaneous volatility σ | 1.5% | 6.5% |
| initial short rate r0 | 5.5% | 5.5% |

This parameter choice generates similar zero-coupon bond prices for the Vasicek and CIR models, which are even almost identical for short and medium maturities up to 5 years.

### 5.1. Comparison results for the AFRM contract with continuous amortization payments

We suppose that $\rho = 5.5\%$ and $A = 1$. For the Vasicek model the analytical approximation formula (4.16) for the optimal prepayment rate is based on the parameters $\alpha = 0.15, \beta = 0.05, \sigma = 0.015$, which yield $R^* = 0.029, \rho^* = 0.0086$ as defined in (4.12) and as stated in Xie et al.(2007a), Figure 4. The corresponding callable mortgage price is calculated with the analytical approximation formula (4.29). The calculation of prices and Greeks according to Section 3 is done with 3 different numbers of binomial steps $N = 100, 500, 1000,$ and allows for a qualitative assessment of the convergence of the binomial trees. Numerical results for smaller maturities up to 5 years are very satisfactory and not listed here. Results for the medium and larger maturities $T = 5, 10, 20$ and 30 years are found in the Tables 5.1-5.3 (Vasicek model) and the Tables 5.4-5.5 (CIR model).

In general, convergence of the binomial tree values to the exact and limiting analytical formulas for the prices of the zero-coupon bond and the non-callable mortgage is excellent. The same observation holds for the Greeks of the non-callable mortgage, where the $\Gamma$ approximations for the CIR model converge better. For the callable mortgage we compare prices and Greeks of mixed binomial analytical values (=difference of analytical values for the non-callable mortgage and binomial tree values for the prepayment option) with pure binomial tree values and obtain a satisfactory convergence, which for $\Gamma$ is again better for the CIR model. A binomial tree approximation of the optimal prepayment rate is obtained following the computational procedure specified after formula (3.8). For the Vasicek model it compares quite favorably with the analytical formula (4.16), which requires unfortunately a tedious and cumbersome analytical determination of its parameters.

It remains to discuss Table 5.3 for the Vasicek model, which compares the analytical approximation of the callable mortgage price by Xie(2009) with its binomial tree counterpart. Compared to the binomial tree values one observes a systematic underestimation (overestimation) of the callable mortgage price (prepayment option price) analytical approximation. This numerical discrepancy increases at the longer maturities and becomes impractical for a valuable estimation of the prepayment option price. Based on these very promising results, we would like to recommend the simplified approach by Costabile and Massabo(2010) for a simultaneous evaluation of mortgage related prices and Greeks for diffusion models of the type (2.10).

**Table 5.1**: Binomial tree vs. exact analytical formulas and limits (Vasicek)

| **model parameters** | α | 0.15 | β | 0.05 | σ | 0.015 | r0 | 0.055 | ρ | 0.055 | A | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| maturity T | | 5 | | | 10 | | | 20 | | | 30 | |
| number of steps N | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 |
| ***zero-coupon bond (100 face value)*** | | | | | | | | | | | | |
| exact analytical price | 76.735 | 76.735 | 76.735 | 59.939 | 59.939 | 59.939 | 37.591 | 37.591 | 37.591 | 23.878 | 23.878 | 23.878 |
| binomial tree price | 76.729 | 76.733 | 76.734 | 59.917 | 59.925 | 59.926 | 37.500 | 37.517 | 37.520 | 23.727 | 23.757 | 23.758 |
| ***non-callable mortgage*** | | | | | | | | | | | | |
| analytical limiting price | 4.3852 | 4.3852 | 4.3852 | 7.7807 | 7.7807 | 7.7807 | 12.561 | 12.561 | 12.561 | 15.582 | 15.582 | 15.582 |
| binomial tree price | 4.3853 | 4.3853 | 4.3853 | 7.7810 | 7.7806 | 7.7806 | 12.560 | 12.558 | 12.558 | 15.574 | 15.569 | 15.569 |
| analytical limiting Δ | -8.3294 | -8.3294 | -8.3294 | -23.328 | -23.328 | -23.328 | -51.308 | -51.308 | -51.308 | -70.896 | -70.896 | -70.896 |
| binomial tree Δ | -8.297 | -8.323 | -8.326 | -23.332 | -23.317 | -23.315 | -51.535 | -51.226 | -51.189 | -71.425 | -70.700 | -70.598 |
| relative absolute deviation | 0.39% | 0.08% | 0.04% | 0.01% | 0.05% | 0.06% | 0.44% | 0.16% | 0.23% | 0.75% | 0.28% | 0.42% |
| analytical limiting Γ | 20.347 | 20.347 | 20.347 | 87.383 | 87.383 | 87.383 | 251.664 | 251.664 | 251.664 | 378.715 | 378.715 | 378.715 |
| binomial tree Γ | 19.710 | 20.178 | 20.237 | 84.655 | 86.018 | 86.179 | 241.538 | 243.857 | 244.225 | 361.303 | 363.989 | 363.894 |
| relative absolute deviation | 3.13% | 0.83% | 0.54% | 3.12% | 1.56% | 1.38% | 4.02% | 3.10% | 2.96% | 4.60% | 3.89% | 3.91% |
| exact analytical Θ | -0.7674 | -0.7674 | -0.7674 | -0.5994 | -0.5994 | -0.5994 | -0.3759 | -0.3759 | -0.3759 | -0.2388 | -0.2388 | -0.2388 |
| binomial tree Θ | -0.7733 | -0.7685 | -0.7679 | -0.6081 | -0.6010 | -0.6002 | -0.3859 | -0.3773 | -0.3763 | -0.2476 | -0.2396 | -0.2386 |
| absolute deviation | 0.59% | 0.12% | 0.06% | 0.87% | 0.16% | 0.08% | 0.99% | 0.14% | 0.04% | 0.88% | 0.08% | 0.02% |

**Table 5.2:** Mixed binomial tree and exact analytical limits (Vasicek)

| **model parameters** | α | 0.15 | β | 0.05 | σ | 0.015 | r0 | 0.055 | ρ | 0.055 | A | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| maturity T | 5 | | | 10 | | | 20 | | | 30 | | |
| number of steps N | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 |
| ***callable mortgage*** | | | | | | | | | | | | |
| mixed bin. anal. price | 4.2864 | 4.322 | 4.3265 | 7.4627 | 7.5284 | 7.5366 | 11.586 | 11.712 | 11.727 | 13.905 | 14.083 | 14.105 |
| binomial tree price | 4.2865 | 4.3221 | 4.3265 | 7.463 | 7.5283 | 7.5364 | 11.585 | 11.708 | 11.723 | 13.897 | 14.071 | 14.092 |
| relative absolute deviation | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.00% | 0.01% | 0.03% | 0.03% | 0.06% | 0.09% | 0.09% |
| mixed bin. anal. Δ | -4.526 | -4.823 | -4.860 | -12.516 | -13.249 | -13.334 | -26.345 | -28.249 | -28.476 | -35.108 | -38.127 | -38.442 |
| binomial tree Δ | -4.494 | -4.816 | -4.856 | -12.519 | -13.238 | -13.320 | -26.573 | -28.167 | -28.358 | -35.637 | -37.932 | -38.145 |
| relative absolute deviation | 0.72% | 0.14% | 0.08% | 0.03% | 0.09% | 0.10% | 0.86% | 0.29% | 0.42% | 1.51% | 0.51% | 0.77% |
| mixed bin. anal. Γ | -210.46 | -200.83 | -199.83 | -406.05 | -382.62 | -381.34 | -649.49 | -585.16 | -577.73 | -790.4 | -678.12 | -667.85 |
| binomial tree Γ | -211.1 | -201.0 | -199.9 | -408.8 | -384.0 | -382.5 | -659.6 | -593.0 | -585.2 | -807.8 | -692.8 | -682.7 |
| relative absolute deviation | 0.30% | 0.08% | 0.06% | 0.67% | 0.36% | 0.32% | 1.56% | 1.33% | 1.29% | 2.20% | 2.17% | 2.22% |
| mixed bin. anal. Θ | -0.7441 | -0.7435 | -0.7433 | -0.5549 | -0.5531 | -0.5527 | -0.3123 | -0.3117 | -0.3116 | -0.1776 | -0.1786 | -0.1783 |
| binomial tree Θ | -0.7501 | -0.7446 | -0.7439 | -0.5636 | -0.5548 | -0.5535 | -0.3222 | -0.3131 | -0.3120 | -0.1864 | -0.1794 | -0.1782 |
| absolute deviation | 0.59% | 0.12% | 0.06% | 0.87% | 0.16% | 0.08% | 0.99% | 0.14% | 0.04% | 0.88% | 0.08% | 0.02% |
| ***prepayment option*** | | | | | | | | | | | | |
| binomial tree price | 0.0988 | 0.0632 | 0.0588 | 0.3181 | 0.2523 | 0.2441 | 0.9753 | 0.8497 | 0.8343 | 1.6767 | 1.4982 | 1.4765 |
| binomial tree Δ | -3.803 | -3.5063 | -3.4692 | -10.813 | -10.079 | -9.9945 | -24.962 | -23.059 | -22.831 | -35.787 | -32.768 | -32.453 |
| binomial tree Γ | 230.81 | 221.18 | 220.17 | 493.43 | 470.00 | 468.73 | 901.15 | 836.82 | 829.4 | 1169.1 | 1056.8 | 1046.6 |
| binomial tree Θ | -0.0232 | -0.0239 | -0.0241 | -0.0445 | -0.0463 | -0.0467 | -0.0636 | -0.0642 | -0.0643 | -0.0612 | -0.0602 | -0.0604 |

**Table 5.3:** Binomial tree vs. analytical approximations (Vasicek)

| **model parameters** | α | 0.15 | β | 0.05 | σ | 0.015 | r0 | 0.055 | ρ | 0.055 | A | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| maturity T | 5 | | | 10 | | | 20 | | | 30 | | |
| number of steps N | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 |
| ***optimal prepayment rate*** | | | | | | | | | | | | |
| analytical approximation | 4.07% | 4.07% | 4.07% | 3.66% | 3.66% | 3.66% | 3.26% | 3.26% | 3.26% | 3.09% | 3.09% | 3.09% |
| binomial tree rate | 4.16% | 4.15% | 4.12% | 3.60% | 3.80% | 3.70% | 3.49% | 3.40% | 3.38% | 3.04% | 3.30% | 3.16% |
| absolute deviation | 0.09% | 0.08% | 0.05% | 0.06% | 0.14% | 0.04% | 0.23% | 0.14% | 0.12% | 0.05% | 0.21% | 0.07% |
| ***callable mortgage*** | | | | | | | | | | | | |
| analytical approximation | 4.3153 | 4.3153 | 4.3153 | 7.4830 | 7.4830 | 7.4830 | 11.559 | 11.559 | 11.559 | 13.841 | 13.841 | 13.841 |
| binomial tree price | 4.2865 | 4.3221 | 4.3265 | 7.463 | 7.5283 | 7.5364 | 11.585 | 11.708 | 11.723 | 13.897 | 14.071 | 14.092 |
| relative absolute deviation | 0.67% | 0.16% | 0.26% | 0.27% | 0.60% | 0.71% | 0.22% | 1.27% | 1.40% | 0.40% | 1.64% | 1.78% |
| ***prepayment option*** | | | | | | | | | | | | |
| analytical approximation | 0.0699 | 0.0699 | 0.0699 | 0.2977 | 0.2977 | 0.2977 | 1.0021 | 1.0021 | 1.0021 | 1.7406 | 1.7406 | 1.7406 |
| binomial tree price | 0.0988 | 0.0632 | 0.0588 | 0.3181 | 0.2523 | 0.2441 | 0.9753 | 0.8497 | 0.8343 | 1.6767 | 1.4982 | 1.4765 |
| absolute deviation | 2.89% | 0.67% | 1.11% | 2.03% | 4.54% | 5.36% | 2.69% | 15.24% | 16.78% | 6.39% | 24.23% | 26.40% |

**Table 5.4**: Binomial tree vs. exact analytical formulas and limits (CIR)

| model parameters | α | 0.15 | β | 0.05 | σ | 0.065 | r0 | 0.055 | ρ | 0.055 | A | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| maturity T | | 5 | | | 10 | | | 20 | | | 30 | |
| number of steps N | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 |
| ***zero-coupon bond (100 face value)*** | | | | | | | | | | | | |
| analytical price formula | 76.735 | 76.735 | 76.735 | 59.906 | 59.906 | 59.906 | 37.389 | 37.389 | 37.389 | 23.551 | 23.551 | 23.551 |
| binomial tree price | 76.730 | 76.734 | 76.735 | 59.899 | 59.905 | 59.906 | 37.389 | 37.393 | 37.393 | 23.554 | 23.559 | 23.558 |
| ***non-callable mortgage*** | | | | | | | | | | | | |
| exact analytical price | 4.3853 | 4.3853 | 4.3853 | 7.7802 | 7.7802 | 7.7802 | 12.549 | 12.549 | 12.549 | 15.542 | 15.542 | 15.542 |
| binomial tree price | 4.3853 | 4.3853 | 4.3853 | 7.7808 | 7.7803 | 7.7803 | 12.554 | 12.551 | 12.550 | 15.553 | 15.546 | 15.544 |
| analytical limiting Δ | -8.280 | -8.280 | -8.280 | -22.940 | -22.940 | -22.940 | -49.500 | -49.500 | -49.500 | -67.572 | -67.572 | -67.572 |
| binomial tree Δ | -8.256 | -8.277 | -8.279 | -23.021 | -22.972 | -22.962 | -50.197 | -49.718 | -49.638 | -69.107 | -68.036 | -67.863 |
| relative absolute deviation | 0.29% | 0.04% | 0.01% | 0.35% | 0.14% | 0.09% | 1.41% | 0.44% | 0.28% | 2.27% | 0.69% | 0.43% |
| analytical limiting Γ | 20.060 | 20.060 | 20.060 | 84.037 | 84.037 | 84.037 | 232.28 | 232.28 | 232.28 | 341.41 | 341.41 | 341.41 |
| binomial tree Γ | 19.462 | 19.932 | 19.993 | 82.359 | 83.671 | 83.842 | 230.22 | 231.89 | 232.09 | 340.36 | 341.38 | 341.46 |
| relative absolute deviation | 2.98% | 0.64% | 0.33% | 2.00% | 0.44% | 0.23% | 0.89% | 0.17% | 0.08% | 0.31% | 0.01% | 0.01% |
| exact analytical Θ | -0.7674 | -0.7674 | -0.7674 | -0.5991 | -0.5991 | -0.5991 | -0.3739 | -0.3739 | -0.3739 | -0.2355 | -0.2355 | -0.2355 |
| binomial tree Θ | -0.7733 | -0.7685 | -0.7679 | -0.6080 | -0.6008 | -0.6000 | -0.3848 | -0.3761 | -0.3750 | -0.2459 | -0.2376 | -0.2366 |
| absolute deviation | 0.59% | 0.12% | 0.06% | 0.89% | 0.18% | 0.09% | 1.09% | 0.22% | 0.11% | 1.03% | 0.21% | 0.11% |

**Table 5.5**: Mixed binomial tree and exact analytical limits (CIR)

| model parameters | α | 0.15 | β | 0.05 | σ | 0.065 | r0 | 0.055 | ρ | 0.055 | A | 1 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| maturity T | | 5 | | | 10 | | | 20 | | | 30 | |
| number of steps N | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 | 100 | 500 | 1000 |
| ***callable mortgage*** | | | | | | | | | | | | |
| mixed bin. anal. price | 4.2856 | 4.3215 | 4.3260 | 7.4607 | 7.5279 | 7.5364 | 11.582 | 11.714 | 11.731 | 13.894 | 14.085 | 14.110 |
| binomial tree price | 4.2857 | 4.3216 | 4.3260 | 7.4613 | 7.5280 | 7.5365 | 11.586 | 11.715 | 11.731 | 13.904 | 14.088 | 14.112 |
| relative absolute deviation | 0.00% | 0.00% | 0.00% | 0.01% | 0.00% | 0.00% | 0.04% | 0.01% | 0.01% | 0.07% | 0.02% | 0.01% |
| mixed bin. anal. Δ | -4.648 | -4.969 | -5.013 | -12.820 | -13.647 | -13.763 | -26.537 | -28.765 | -29.080 | -34.871 | -38.367 | -38.843 |
| binomial tree Δ | -4.624 | -4.966 | -5.012 | -12.900 | -13.679 | -13.784 | -27.235 | -28.983 | -29.219 | -36.406 | -38.832 | -39.135 |
| relative absolute deviation | 0.52% | 0.06% | 0.02% | 0.63% | 0.23% | 0.16% | 2.63% | 0.76% | 0.48% | 4.40% | 1.21% | 0.75% |
| mixed bin. anal. Γ | -206.9 | -196.7 | -195.6 | -395.8 | -371.8 | -370.0 | -635.9 | -567.0 | -558.5 | -776.8 | -660.6 | -648.4 |
| binomial tree Γ | -207.5 | -196.8 | -195.6 | -397.5 | -372.2 | -370.2 | -637.9 | -567.4 | -558.7 | -777.8 | -660.7 | -648.4 |
| relative absolute deviation | 0.29% | 0.07% | 0.03% | 0.42% | 0.10% | 0.05% | 0.32% | 0.07% | 0.03% | 0.14% | 0.00% | 0.01% |
| mixed bin. anal. Θ | -0.7445 | -0.7433 | -0.7432 | -0.5549 | -0.5533 | -0.5530 | -0.3120 | -0.3120 | -0.3120 | -0.1769 | -0.1783 | -0.1782 |
| binomial tree Θ | -0.7504 | -0.7445 | -0.7438 | -0.5638 | -0.5550 | -0.5539 | -0.3229 | -0.3142 | -0.3131 | -0.1872 | -0.1804 | -0.1793 |
| absolute deviation | 0.59% | 0.12% | 0.06% | 0.89% | 0.18% | 0.09% | 1.09% | 0.22% | 0.11% | 1.03% | 0.21% | 0.11% |
| ***prepayment option*** | | | | | | | | | | | | |
| binomial tree price | 0.0996 | 0.0637 | 0.0593 | 0.3194 | 0.2523 | 0.2438 | 0.9679 | 0.8355 | 0.8187 | 1.6484 | 1.4576 | 1.4324 |
| binomial tree Δ | -3.632 | -3.311 | -3.267 | -10.120 | -9.293 | -9.178 | -22.962 | -20.734 | -20.419 | -32.701 | -29.205 | -28.728 |
| binomial tree Γ | 226.94 | 216.76 | 215.62 | 479.85 | 455.86 | 454.03 | 868.17 | 799.27 | 790.82 | 1118.2 | 1002.1 | 989.83 |
| binomial tree Θ | -0.0229 | -0.0240 | -0.0242 | -0.0441 | -0.0458 | -0.0461 | -0.0619 | -0.0619 | -0.0619 | -0.0586 | -0.0573 | -0.0573 |
| ***optimal prepayment rate*** | | | | | | | | | | | | |
| binomial tree rate | 4.20% | 4.20% | 4.18% | 3.70% | 3.89% | 3.81% | 3.59% | 3.55% | 3.54% | 3.19% | 3.45% | 3.35% |

## 5.2. Illustration for the AVRM contract with discrete amortization payments

First of all, some methodological and practical aspects must be discussed. As defined in Section 2.2, the deterministic contract rates do not depend on the future stochastic evolution of the short rates $r_s$, $s = 0,...,T-1$, that follow the Vasicek/CIR models. The contract rate structure depends upon the current market bond prices or equivalently on the current TSIR, which is used to estimate implied forward market mortgage equivalent rates in accordance with formula (2.19). Of course, the current market bond prices, which determine the contract rates, are inconsistent with the Vasicek/CIR bond prices because these one-factor models cannot reproduce in general the current TSIR whatever the choice of the parameters. However, fixing contract agreements based on the current state of the world (pricing activity) and valuation of contract features based on the unknown future using stochastic models (risk management activity) are not contradictory per se (they correspond to different activities within the organization of a financial institution). If current market bond prices must be in line with the interest rate models, then more complex models must be considered; e.g. the yield curve fitting models by Hull and White(1990b/2001), Black et al.(1990), Black and Karinski(1991), or the LIBOR market model by Brace et al.(1997), Jamshidan(1997)) and Miltersen et al.(1997). This important distinction is illustrated in Table 5.6, which displays the possible numerical differences between market forward prices and modeled forward prices. Note that our choice of market forward bond prices is arbitrary (it only fulfills the purpose of illustration) and does not rely on real-world forward bond prices. In practice, the latter are often derived from zero coupon bond yield curves published by national banks (e.g. "Statistisches Monatsheft der Schweizerischen Nationalbank", available at www.snb.ch). Our deliberately simple AVRM contract definition makes it a path-independent financial instrument that can be valued with the same convenient backward recursion formula (3.8) as used for the AFRM contract. This contract is enough flexible to fit its variable deterministic contract rate structure to the current forward bond price structure. With this property, a valuable practical alternative to the AFRM contract has been introduced and motivated.

In contrast, the known AVRM contract from the literature is a path-dependent interest rate instrument (e.g. Sharp(2006)). This means that the contract rates are contingent on the historical evolution of the interest rates. In this situation, the amortization payments, the outstanding loan balance values and the face values in (2.20)-(2.22) will depend upon the lattice that describes the interest rate evolution involved in the backward recursion valuation formula (2.16). As a consequence the simple recursion (3.8) must be replaced by an algorithm of exponential time to maturity complexity (e.g. Hochreiter and Pflug(2006), Section 2.2). To reduce the involved computational complexity, it is common to use an auxiliary state variable (e.g. Hull and White (1993), Ritchken et al.(1993), Willmot et al.(1993), Barraquand and Pudet(1996)). Another promising approach is the "just-in-time" least squares Monte-Carlo method proposed in Dutte and Welke(2008), which includes the Vasicek/CIR models. The idea of this method consists to start from the final interest rate distribution and generate stochastic interest rates backwards as the mortgage prepayment option is priced. Some desirable objectives can be achieved this way:

(i) The backward option pricing algorithm is in line with the backward interest rate process.
(ii) The storage requirement is greatly reduced, which results in an increased efficiency.
(iii) The MC errors can be reduced, which results in an increased accuracy.

In our numerical illustration we set again $A = 1$. The adjustment in (2.19) is done with $teaser = 0.01$, periodic cap and floor $y = 0.005$ and lifetime cap $\ell = 0.02 \cdot T/10$. For comparison purposes the initial approximate continuous contract rate is set equal to the initial short rate of the AFRM contract, i.e. $\rho(0) = r_0 = 5.5\%$. To fulfill the latter condition we use a variable margin defined by $margin = teaser + (e^{r_0 h} - 1)/h - IF^F(0)$. The calculation of prices and Greeks follows again Section 3. It is done with a fixed number of steps $N = 1200$. We vary $IP = 1/4, 1/12, h$ over the maturity years $T = 5, 10, 20$. Results are summarized in the Table 5.7 (Vasicek model) and the Table 5.8 (CIR model). It is interesting to note that the results for the limiting case $IP = h$ (numerical approximation of the continuous payment case) are similar to those of Section 5.1. The higher prices are due to the yearly adjustment of the contract rates. The special case $y = 0$ of the extended AVRM algorithm reduces to an AFRM algorithm. Indeed, in this situation the variable contract rates (2.19) are all equal, i.e. $\rho_D(s, r_s) = \rho_D(0, r_0)$, $s = 1,...,T-1$. One observes that the prepayment option prices and their Greeks vary monotonically with the cycle length $IP$. The binomial tree $\Theta$'s of the non-callable and callable mortgages for discrete $IP = 1/4, 1/12$ and continuous approximation $IP = h$ have different signs. This is due to the fact that the non-callable market values in the formula (3.11) are not influenced by cash-flow payments for these discrete values while they are for the continuous approximation. Finally, for further information, the Table 5.9 displays the dependence of the discrete amortization payments $A_P(s)$, that have been defined in (2.20), of the various AVRM contracts upon the different interest payment cycles.

**Table 5.6:** Market forward bond prices versus Vasicek and CIR forward bond prices

| model and parameters | Market TSIR | | | Vasicek TSIR | α 0.15, β 0.05, σ 0.015, r0 0.055 | | CIR TSIR | α 0.15, β 0.05, σ 0.065, r0 0.055 | |
|---|---|---|---|---|---|---|---|---|---|
| maturity T | 5 | 10 | 20 | 5 | 10 | 20 | 5 | 10 | 20 |
| time s | Market forward bond prices | | | Vasicek forward bond prices | | | CIR forward bond prices | | |
| 0 | 0.76735 | 0.59900 | 0.37600 | 0.76735 | 0.59939 | 0.37591 | 0.76735 | 0.59906 | 0.37389 |
| 1 | 0.81042 | 0.63300 | 0.39700 | 0.81042 | 0.63303 | 0.39701 | 0.81042 | 0.63268 | 0.39487 |
| 2 | 0.85520 | 0.66800 | 0.41900 | 0.85520 | 0.66801 | 0.41895 | 0.85520 | 0.66764 | 0.41669 |
| 3 | 0.90170 | 0.70400 | 0.44200 | 0.90170 | 0.70433 | 0.44173 | 0.90170 | 0.70394 | 0.43935 |
| 4 | 0.94995 | 0.74200 | 0.46500 | 0.94995 | 0.74202 | 0.46537 | 0.94994 | 0.74161 | 0.46286 |
| 5 | | 0.78100 | 0.49000 | | 0.78111 | 0.48989 | | 0.78069 | 0.48725 |
| 6 | | 0.82200 | 0.51500 | | 0.82167 | 0.51532 | | 0.82124 | 0.51256 |
| 7 | | 0.86400 | 0.54200 | | 0.86375 | 0.54172 | | 0.86336 | 0.53885 |
| 8 | | 0.90700 | 0.56900 | | 0.90744 | 0.56912 | | 0.90713 | 0.56616 |
| 9 | | 0.95300 | 0.59800 | | 0.95283 | 0.59758 | | 0.95264 | 0.59457 |
| 10 | | | 0.62700 | | | 0.62717 | | | 0.62413 |
| 11 | | | 0.65800 | | | 0.65793 | | | 0.65491 |
| 12 | | | 0.69000 | | | 0.68995 | | | 0.68699 |
| 13 | | | 0.72300 | | | 0.72328 | | | 0.72045 |
| 14 | | | 0.75800 | | | 0.75801 | | | 0.75535 |
| 15 | | | 0.79400 | | | 0.79420 | | | 0.79179 |
| 16 | | | 0.83200 | | | 0.83194 | | | 0.82984 |
| 17 | | | 0.87100 | | | 0.87130 | | | 0.86960 |
| 18 | | | 0.91200 | | | 0.91237 | | | 0.91114 |
| 19 | | | 0.95500 | | | 0.95524 | | | 0.95458 |

**<u>Table 5.7</u>:** Prices and Greeks for the AVRM contract with given IP (Vasicek model)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **model parameters** | α | 0.15 | β | 0.05 | σ | 0.015 | r0 | 0.055 | |
| periodic cap and floor y | 0.005 | teaser | 0.01 | | | | | | |
| margin | 0.011692 | 0.011927 | 0.012038 | 0.013434 | 0.013654 | 0.013752 | 0.015817 | 0.016017 | 0.016097 |
| maturity T | | 5 | | | 10 | | | 20 | |
| number of steps N | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 |
| cycle length IP | 0.25 | 0.083 | 0.004 | 0.25 | 0.083 | 0.008 | 0.25 | 0.083 | 0.017 |
| parameter nI = IP·N/T | 60 | 20 | 1 | 30 | 10 | 1 | 15 | 5 | 1 |
| lifetime cap ℓ | 0.01 | 0.01 | 0.01 | 0.02 | 0.02 | 0.02 | 0.04 | 0.04 | 0.04 |
| ***non-callable mortgage*** | | | | | | | | | |
| binomial tree price | 4.432 | 4.433 | 4.433 | 7.993 | 7.997 | 8.000 | 13.266 | 13.288 | 13.296 |
| binomial tree Δ | -8.830 | -8.566 | -8.441 | -24.576 | -24.225 | -24.066 | -54.906 | -54.568 | -54.431 |
| binomial tree Γ | 21.873 | 20.969 | 20.544 | 91.488 | 89.796 | 89.039 | 262.437 | 260.515 | 259.747 |
| binomial tree Θ | 0.235 | 0.235 | -0.765 | 0.411 | 0.412 | -0.589 | 0.660 | 0.661 | -0.340 |
| ***callable mortgage*** | | | | | | | | | |
| binomial tree price | 4.111 | 4.268 | 4.341 | 7.386 | 7.537 | 7.603 | 11.723 | 11.869 | 11.927 |
| binomial tree Δ | -2.869 | -3.479 | -3.848 | -8.554 | -9.375 | -9.749 | -18.662 | -19.394 | -19.681 |
| binomial tree Γ | -182.360 | -221.467 | -212.945 | -443.059 | -449.570 | -448.830 | -791.321 | -784.665 | -782.895 |
| binomial tree Θ | 0.244 | 0.257 | -0.741 | 0.450 | 0.458 | -0.539 | 0.720 | 0.727 | -0.272 |
| ***prepayment option*** | | | | | | | | | |
| binomial tree price | 0.322 | 0.165 | 0.092 | 0.606 | 0.461 | 0.396 | 1.544 | 1.419 | 1.369 |
| binomial tree Δ | -5.961 | -5.088 | -4.593 | -16.021 | -14.849 | -14.316 | -36.244 | -35.174 | -34.750 |
| binomial tree Γ | 204.233 | 242.435 | 233.490 | 534.547 | 539.366 | 537.868 | 1053.758 | 1045.181 | 1042.642 |
| binomial tree Θ | -0.010 | -0.022 | -0.025 | -0.039 | -0.046 | -0.049 | -0.061 | -0.066 | -0.068 |
| ***optimal prepayment rate*** | | | | | | | | | |
| binomial tree rate | 5.11% | 4.53% | 4.34% | 4.68% | 4.40% | 4.13% | 4.14% | 3.95% | 3.95% |

**<u>Table 5.8</u>:** Prices and Greeks for the AVRM contract with given IP (CIR model)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **model parameters** | α | 0.15 | β | 0.05 | σ | 0.065 | r0 | 0.055 | |
| periodic cap and floor y | 0.005 | teaser | 0.01 | | | | | | |
| margin | 0.011692 | 0.011927 | 0.012038 | 0.013434 | 0.013654 | 0.013752 | 0.015817 | 0.016017 | 0.016097 |
| maturity T | | 5 | | | 10 | | | 20 | |
| number of steps N | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 |
| cycle length IP | 0.25 | 0.083 | 0.004 | 0.25 | 0.083 | 0.008 | 0.25 | 0.083 | 0.017 |
| parameter nI = IP·N/T | 60 | 20 | 1 | 30 | 10 | 1 | 15 | 5 | 1 |
| lifetime cap ℓ | 0.01 | 0.01 | 0.01 | 0.02 | 0.02 | 0.02 | 0.04 | 0.04 | 0.04 |
| ***non-callable mortgage*** | | | | | | | | | |
| binomial tree price | 4.432 | 4.433 | 4.433 | 7.992 | 7.997 | 7.999 | 13.258 | 13.280 | 13.289 |
| binomial tree Δ | -8.778 | -8.517 | -8.393 | -24.196 | -23.854 | -23.699 | -53.228 | -52.905 | -52.775 |
| binomial tree Γ | 21.603 | 20.714 | 20.298 | 88.982 | 87.357 | 86.629 | 249.411 | 247.621 | 246.905 |
| binomial tree Θ | 0.235 | 0.235 | -0.765 | 0.411 | 0.412 | -0.589 | 0.661 | 0.663 | -0.338 |
| ***callable mortgage*** | | | | | | | | | |
| binomial tree price | 4.109 | 4.266 | 4.340 | 7.380 | 7.531 | 7.598 | 11.708 | 11.855 | 11.914 |
| binomial tree Δ | -3.086 | -3.698 | -4.059 | -9.262 | -10.062 | -10.422 | -20.118 | -20.834 | -21.106 |
| binomial tree Γ | -188.284 | -221.092 | -210.804 | -445.756 | -446.666 | -443.257 | -783.772 | -771.665 | -768.161 |
| binomial tree Θ | 0.246 | 0.258 | -0.740 | 0.451 | 0.459 | -0.539 | 0.720 | 0.726 | -0.272 |
| ***prepayment option*** | | | | | | | | | |
| binomial tree price | 0.323 | 0.167 | 0.093 | 0.612 | 0.466 | 0.402 | 1.550 | 1.425 | 1.375 |
| binomial tree Δ | -5.692 | -4.819 | -4.334 | -14.934 | -13.792 | -13.277 | -33.110 | -32.072 | -31.669 |
| binomial tree Γ | 209.886 | 241.806 | 231.102 | 534.737 | 534.023 | 529.886 | 1033.183 | 1019.286 | 1015.066 |
| binomial tree Θ | -0.011 | -0.023 | -0.025 | -0.040 | -0.047 | -0.049 | -0.059 | -0.064 | -0.066 |
| ***optimal prepayment rate*** | | | | | | | | | |
| binomial tree rate | 5.11% | 4.56% | 4.38% | 4.69% | 4.19% | 4.07% | 4.20% | 4.03% | 3.86% |

**Table 5.9:** Dependence of discrete amortization payments $A_P(s)$ upon interest payment cycle

| maturity T | 5 | | | 10 | | | 20 | | |
|---|---|---|---|---|---|---|---|---|---|
| number of steps N | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 | 1200 |
| cycle length IP | 0.25 | 0.083 | 0.004 | 0.25 | 0.083 | 0.008 | 0.25 | 0.083 | 0.017 |
| time s | | | | | | | | | |
| 0 | 0.25149 | 0.08350 | 0.00417 | 0.25130 | 0.08348 | 0.00834 | 0.25099 | 0.08346 | 0.01667 |
| 1 | 0.25405 | 0.08432 | 0.00421 | 0.25661 | 0.08523 | 0.00851 | 0.26101 | 0.08678 | 0.01734 |
| 2 | 0.25570 | 0.08485 | 0.00423 | 0.26065 | 0.08655 | 0.00864 | 0.26965 | 0.08965 | 0.01791 |
| 3 | 0.25559 | 0.08481 | 0.00423 | 0.26039 | 0.08647 | 0.00863 | 0.26907 | 0.08946 | 0.01787 |
| 4 | 0.25552 | 0.08479 | 0.00423 | 0.26008 | 0.08637 | 0.00862 | 0.26876 | 0.08936 | 0.01785 |
| 5 | | | | 0.25989 | 0.08631 | 0.00862 | 0.26825 | 0.08919 | 0.01782 |
| 6 | | | | 0.25966 | 0.08623 | 0.00861 | 0.26799 | 0.08910 | 0.01780 |
| 7 | | | | 0.25955 | 0.08620 | 0.00861 | 0.26755 | 0.08896 | 0.01777 |
| 8 | | | | 0.25957 | 0.08621 | 0.00861 | 0.26737 | 0.08890 | 0.01776 |
| 9 | | | | 0.25946 | 0.08618 | 0.00860 | 0.26704 | 0.08879 | 0.01774 |
| 10 | | | | | | | 0.26697 | 0.08877 | 0.01774 |
| 11 | | | | | | | 0.26677 | 0.08870 | 0.01772 |
| 12 | | | | | | | 0.26665 | 0.08866 | 0.01772 |
| 13 | | | | | | | 0.26661 | 0.08865 | 0.01771 |
| 14 | | | | | | | 0.26648 | 0.08861 | 0.01770 |
| 15 | | | | | | | 0.26645 | 0.08860 | 0.01770 |
| 16 | | | | | | | 0.26637 | 0.08857 | 0.01770 |
| 17 | | | | | | | 0.26639 | 0.08858 | 0.01770 |
| 18 | | | | | | | 0.26640 | 0.08858 | 0.01770 |
| 19 | | | | | | | 0.26640 | 0.08858 | 0.01770 |

**Acknowlegements.** The author is very grateful to the referees for careful reading, corrections, critical comments and helpful suggestions for improved presentation.

## References

*Agarwal, S., Driscoll, J.C. and D.I. Laibson* (2008). Optimal mortgage refinancing: A closed form solution. NBER Working Paper no. 13487. URL: http://ssrn.com/abstract=1010702

*Barraquand, J. and T. Pudet* (1996). Pricing of American Path Dependent Contingent Claims. Journal of Finance 6, 17-51.

*Black, F., Derman, E. and W. Toy* (1990). A one factor model of interest rates and its application to treasury bond options. Financial Analysts Journal 46, 33–39.

*Black, F. and P. Karasinski* (1991). Bond and option pricing when short rates are lognormal. Financial Analysis Journal 47, 52–59.

*Belomestny*, D., *Milstein, G.* and *J. Schoenmakers* (2007). Sensitivities for Bermudan options by regression methods. Preprint WIAS , no. 1247, Berlin.

*Bian, B., Jiang, L. and F. Yi* (2005). A parabolic variational inequality arising from the valuation of fixed rate mortgage. European Journal of Applied Mathematics 16(3), 361-383.

*Björk, T.* (1998). Arbitrage Theory in Continuous Time. Oxford University Press, Oxford.

*Brace, A., Gatarek, D. and M. Musiela* (1997). The market model of interest rate dynamics. Mathematical Finance 7(2), 127–155.

*Brigo, D. and F. Mercurio* (2001). On deterministic-shift extensions of short-rate models. Finance and Stochastics 5, 369-387.

*Büttler, H.-J. and J. Waldvogel* (1996). Pricing callable bonds by means of Green's function. Mathematical Finance 6(1), 53-88.

*Chen, R.R. and L. Scott* (2003). Multi-factor Cox-Ingersoll-Ross models of the term structure: estimates and tests from a Kalman filter model. J. Real Estate Fin. Econ. 27(2), 143-172.

*Chen, Y., Connolly, M., Tang, W. and T. Su* (2009). The value of mortgage prepayment and default options. The Journal of Futures Markets 29(9), 840-861.

*Chung, S.-L. and M. Shackelton* (2002). The binomial Black-Scholes model and the Greeks. The Journal of Future Markets 22(2), 143-153.

*Costabile, M., Leccadito, A. and I. Massabo* (2009). Computationally simple lattice methods for option and bond pricing. Decisions in Economics and Finance 32, 161-181.

*Costabile, M. and I. Massabo* (2010). A simplified approach to approximate diffusion processes widely used in finance. Journal of Derivatives 17(3), 65-85.

*Cox, J.C., Ross, S.A. and M. Rubinstein* (1979). Option pricing: a simplified approach. Journal of Financial Economics 7, 229-263.

*Cox, J.C. and M. Rubinstein* (1985). Option Markets. Englewood Cliffs, NJ: Prentice Hall.

*Cox, J.C., Ingersoll, J.E. and S.A. Ross* (1985). A theory of the term structure of interest rates. Econometrica 53, 385-402.

*De Rossi, G. and T. Vargiolu* (2010). Optimal prepayment and default rules for mortgage-backed securities. Decisions in Economics and Finance 33(1), 23-47.

*De Rozario, R.* (2004). New methods for calculating Greeks in binomial tree models. School of Banking and Finance, University of NSW, Sydney.

*Dickinson, A. and A.J. Heuson* (1994). Mortgage prepayments: past and present. Journal of Real Estate Literature 2, 11-33.

*Dutte, S.K. and G.M. Welke* (2008). Just-in-time Monte Carlo for path dependent American options. The Journal of Derivatives 15(4), 29-47.

*Duffy, D.J.* (2006). Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach. Wiley, New York.

*Goncharov, Y.* (2003). Mathematical Theory of Mortgage Modeling. Ph.D. Thesis, University of Illinois, Chicago.

*Goncharov, Y.* (2004). On revision of the option-based approach to modeling mortgage securities. Proceedings of Computational Finance Conference, Bologna, Italy.

*Goncharov, Y.* (2005). On efficient implementation of option-based approach to mortgage modeling. Preprint. URL: http://www.math.fsu.edu/~goncharo/.

*Hall, A.R.* (1985). Valuing the mortgage borrower's prepayment option. Real Estate Economics 13(3), 229-247.

*Hochreiter, R. and G.Ch. Pflug* (2006). Polynomial algorithms for pricing path-dependent interest rate financial instruments. Journal Computational Economics 28(3), 291-309.

*Hull, J.* (1993/2003). Options, futures and other derivatives (2nd and 5th edition). Prentice Hall.

*Hull, J. and A. White* (1990a). Valuing derivative securities using the explicit finite difference method. J. of Financial and Quantitative Analysis 25, 87-100.

*Hull, J. and A. White* (1990b). Pricing interest rate derivatives securities. Review of Financial Studies 3(4), 573-592.

*Hull, J. and A. White* (1993). Efficient procedures for valuing European and American path-dependent options. Journal of Derivatives 1(1), 21–31.

*Hull, J. and A. White* (2001). The general Hull-White model and super calibration. Financial Analysts Journal 57(6), 34-43.

*Jamshidan, F.* (1997). LIBOR and swap market models and measures. Finance and Stochastics 1(4), 293–330.

*Jeffrey, A. and D. Zwillinger* (2000). Table of Integrals, Series, and Products. 6$^{th}$ Ed. Elsevier, Amsterdam.

*Kalotay, A.J., Williams, G.O. and Fabozzi, F.J.* (2004). An option-theoretic prepayment model for mortgages and mortgage-backed securities. Int. J. Theor. Appl. Finance 7, 949-978.

*Kau. J.B. and D.C. Keenan* (1995). An overview of the option-theoretic pricing of mortgages. Journal of Housing Research 6(2), 217-244.

*Kau. J.B., Keenan, D.C., Muller, W.J.III and J.F. Epperson* (1992). A generalized valuation model for fixed-rate residential mortgages. J. of Money, Credit & Banking 24, 279-299.

*Kau. J.B., Keenan, D.C., Muller, W.J.III and J.F. Epperson* (1993). Option theory and floating-rate securities with a comparison of adjustable- and fixed-rate mortgages. Journal of Business 66, 595-618.

*Lo, C.F., Lau C.S. and C.H. Hui* (2009). Valuation of fixed rate mortgages by moving boundary approach. Proc. of the World Congress on Engineering (WCE), Vol II, London, 6 p.

*Longstaff, F.A. and E. Schwartz* (2001). Valuing American options by simulation: a simple least squares approach. Review of Financial Studies 14, 113-148.

*Miltersen, K., Sandmann, K. and D. Sondermann* (1997). Closed Form Solutions for Term Structure Derivatives with Log Normal Interest Rates. Journal of Finance 52(1), 409-430.

*Nelson, D.B. and K. Ramaswamy* (1990). Simple binomial processes as diffusion approximations in financial models. Review of Financial Studies 3(3), 393-430.

*Ritchken, P., Sankarasubramanian, L. and A. Vijh* (1993).The Valuation of Path Dependent Contracts on the Average. Management Science 39, 1202–1213.

*Schönbucher, Ph. J.* (2000). Credit risk modeling and credit derivatives. Ph.D. Thesis, Bonn.

*Schwartz, E. and W. Torous* (1991). Caps on adjustable rate mortgages: valuation, insurance and hedging. In: Hubbard, R.G.(Ed.). Financial Markets and Financial Crises, 283-304. University of Chicago Press, Chicago.

*Schwartz, E. and W. Torous* (1992). Prepayment, default and the valuation of mortgage pass-through securities. Journal of Business 65(2), 221-239.

*Sharp, N.J.* (2006). Advances in mortgage valuation: an option-theoretic approach. Ph.D. Thesis, School of Mathematics, University of Manchester.

*Sharp, N.J., Duck, P.A. and D.P. Newton* (2008). An improved fixed-rate mortgage valuation methodology with interacting prepayment and default options. J. Real Estate Fin. Econ. 36(3), 307-342.

*Sharp, N.J., Johnson, P.V., Newton, D.P. and P.A. Duck* (2009). A new prepayment model with default: an Occupation-time derivative approach. J. Real Estate Fin. Econ. 39(2), 118-145.

*Siegel, J.J.* (1984). The Mortgage Refinancing Decision. Housing Finance Review 3, 91-97.

*Stanton, R.* (1995). Rational prepayment and the valuation of mortgage-backed securities. The Review of Financial Studies 8, 677-708.

*Stanton, R. and N. Wallace* (1995). ARM wrestling: valuing adjustable rate mortgages indexed to the eleventh district cost of funds. Real Estate Economics 23, 311-345.

*Tavella, D. and C. Randall* (2000). Pricing Financial Instruments: The Finite Difference Method. Wiley, New York.

*Vasicek, O.A.* (1977). An equilibrium characterization of the term structure. J. of Financial Economics 5, 177-188.

*Wallner, Ch. and U. Wystup* (2004). Efficient computation of option price sensitivities for options of American style. Willmot Magazine, November Issue, 72-81.

*Wilmott, P., Dewynne, J. and S. Howison* (1993). Option Pricing: Mathematical Models and Computation. Oxford Financial Press, Oxford.

*Xie, D., Chen, X. and J. Chadam* (2007a). Optimal payment of mortgages. European Journal of Applied Mathematics 18, 363-388.

*Xie, D., Chen, X. and J. Chadam* (2007b). Numerical solution to a free boundary problem arising from mortgage pricing. World Congress Engineering Computer Science (WCECS), 5 p.

*Xie, D.* (2008). Numerical valuation of fixed rate mortgages. IAENG International Journal of Applied Mathematics 38(2), 10 p.

*Xie, D.* (2009). Fixed rate mortgages: valuation and closed form approximations. IAENG International Journal of Applied Mathematics 39(1), 9 p.

*Xie, D., Edwards, D. and G. Schleiniger* (2010). An Asymptotic Method to a Financial Optimization Problem. In: S.-I. Ao, B. Rieger and M. Amouzegar (Eds.). Advances in Machine Learning and Data Analysis. LN Electrical Engineering 48, 79-94. Springer, Berlin.