

## Chapter Five

# Amortization Method and Sinking Funds

### Section 5.1

### Amortization of a Debt

In this chapter we shall discuss different methods of repaying interest-bearing loans, which is one of the most important applications of annuities in business transactions.

The first and most common method is the **amortization method**. When this method is used to liquidate an interest-bearing debt, a series of periodic payments, usually equal, is made. Each payment pays the interest on the unpaid balance and also repays a part of the outstanding principal. As time goes on, the outstanding principal is gradually reduced and interest on the unpaid balance decreases.

When a debt is amortized by equal payments at equal payment intervals, the debt becomes the discounted value of an annuity. The size of the payment is determined by the methods used in the annuity problems of the preceding chapters. The common commercial practice is to round the payment up to the next cent. This practice will be used in this textbook unless specified otherwise. Instead of rounding up to the next cent, the lender may round up to the next dime or dollar. In any case the rounding of the payment up to the cent, to the dime or to the dollar will result in a smaller concluding payment. An equation of value at the time of the last payment will give the size of the smaller concluding payment.

**EXAMPLE 1** A loan of \$20 000 is to be amortized with equal monthly payments over a period of 10 years at  $j_{12} = 12\%$ . Find the concluding payment if the monthly payment is rounded up to a) the cent; b) the dime.

**Solution** First we find the monthly payment  $R$ , given  $A = 20\,000$ ,  $n = 120$  and  $i = .01$ . Using equation (11) of Section 3.3 we calculate

$$R = \frac{20\,000}{a_{\overline{120}|.01}} = 286.9418968$$



and set up an equation of value for  $X$  at 120,

$$X + R s_{\overline{119}|0.01}(1.01) = 20\,000(1.01)^{120}$$

**Solution a** If the monthly payment is rounded up to the next cent, we have  $R = \$286.95$  and calculate

$$\begin{aligned} X &= 20\,000(1.01)^{120} - 286.95 s_{\overline{119}|0.01}(1.01) \\ &= 66\,007.74 - 65\,722.65 \\ &= \$285.09 \end{aligned}$$

**Solution b** If the monthly payment is rounded up to the next dime, we have  $R = \$287$  and calculate

$$\begin{aligned} X &= 20\,000(1.01)^{120} - 287 s_{\overline{119}|0.01}(1.01) \\ &= 66\,007.74 - 65\,734.10 \\ &= \$273.64 \end{aligned}$$

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When interest-bearing debts are amortized by means of a series of equal payments at equal intervals, it is important to know how much of each payment goes for interest and how much goes for the reduction of principal. For example, this may be a necessary part of determining one's taxable income or tax deductions. We construct an **amortization schedule**, which shows the progress of the amortization of the debt.

**CALCULATION TIP:** The common commercial practice for loans that are amortized by equal payments is to round the payment up to the next cent (sometimes up to the next nickel, dime or dollar if so specified). For each loan you should calculate the reduced final payment. For all other calculations (other than the regular amortization payment), use the normal round-off procedure. Do not round off the interest rate per conversion period. Use all the digits provided by your calculator (store the rate in a memory of the calculator) to avoid significant round-off errors. When calculating the total payout of the loan (also called the total debt in the sum of digits method of section 5.4), find the total sum of all regular (rounded-up) payments and the final reduced (rounded-off) payment.

**EXAMPLE 2** A debt of \$22 000 with interest at  $j_4 = 10\%$  is to be amortized by payments of \$5000 at the end of each quarter for as long as necessary. Make out an amortization schedule showing the distribution of the payments as to interest and the repayment of principal.

**Solution** The interest due at the end of the first quarter is  $2\frac{1}{2}\%$  of \$22 000 or \$550.00. The first payment of \$5000 at this time will pay the interest and will also reduce the **outstanding principal balance\*** by \$4450. Thus the outstanding principal after the first payment is reduced to \$17 550. The interest due at the end of the second quarter is  $2\frac{1}{2}\%$  of \$17 550 or \$438.75. The second payment of \$5000 pays the interest and reduces the indebtedness by \$4561.25. The outstanding principal now becomes \$12 988.75. This procedure is repeated and the results are tabulated below in the amortization schedule.

Payment Number	Periodic Payment	Payment of Interest at $2\frac{1}{2}\%$	Principal Repaid	Outstanding Balance
				\$22 000.00
1	5 000.00	550.00	4 450.00	17 550.00
2	5 000.00	438.75	4 561.25	12 988.75
3	5 000.00	324.72	4 675.28	8 313.47
4	5 000.00	207.84	4 792.16	3 521.31
5	3 609.34	88.03	3 521.31	0
Totals	23 609.34	1609.34	22 000.00	

It should be noted that the 5th payment is only \$3609.34, which is the sum of the outstanding principal at the end of the 4th quarter plus the interest due at  $2\frac{1}{2}\%$ . The totals at the bottom of the schedule are for checking purposes. The total amount of principal repaid must equal the original debt. Also, the total of the periodic payments must equal the total interest plus the total principal returned. Note that the entries in the principal repaid column (except the final payment) are in the ratio  $(1 + i)$ . That is

$$\frac{4561.25}{4450.00} \doteq \frac{4675.28}{4561.25} \doteq \frac{4792.16}{4675.28} \doteq 1.025.$$

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**EXAMPLE 3** Consider a \$6000 loan that is to be repaid over 5 years with monthly payments at  $j_{12} = 6\%$ . Show the first three lines of the amortization schedule.

**Solution** First, we find the monthly payment  $R$  given  $A = 6000$ ,  $n = 60$  and  $i = .005$ .

$$R = \frac{6000}{a_{\overline{60}|.005}} = \$116.00$$

To make out the amortization schedule for the first three months, we follow the method outlined in Example 2 with  $i = .005$ .

\**Outstanding principal balance* is also referred to as outstanding principal or outstanding balance.

Payment Number	Periodic Payment	Payment of Interest at $i$	Principal Repaid	Outstanding Balance
				\$6000.00
1	116.00	30.00	86.00	5914.00
2	116.00	29.57	86.43	5827.57
3	116.00	29.14	86.86	5740.71

As before, the entries in the principal repaid column are in the ratio  $1 + i$ . That is,

$$\frac{86.43}{86.00} \doteq \frac{86.86}{86.43} \doteq 1.005$$

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**EXAMPLE 4** Prepare a spreadsheet and show the first 3 months and the last 3 months of a complete amortization schedule for the loan of Example 3 above. Also show total payments, total interest and total principal paid.

**Note:** There are several spreadsheets available (such as EXCEL, LOTUS, QUATTRO PRO) and all can generate required schedules very well. In this text we will present Excel spreadsheets.

**Solution** We summarize the entries in an Excel spreadsheet and their interpretation in the table below. In any amortization schedule we reserve cells A1 through E1 for headings. The same headings will be used in all future Excel spreadsheets showing amortization schedules.

	CELL	ENTER	INTERPRETATION
Headings	A1	Pmt #	'Payment number'
	B1	Payment	'Monthly payment'
	C1	Interest	'Interest payment'
	D1	Principal	'Principal payment'
	E1	Balance	'Outstanding principal balance'
Line 0	A2	0	Time starts
	E2	6,000	Mortgage amount
	A3	=A2+1	End of month 1
Line 1	B3	116.00	Monthly payment (up to the next cent)
	C3	=E2*0.005	Interest 1 = (Balance 0) (0.005)
	D3	=B3-C3	Principal 1 = Payment - Interest 1
	E3	=E2-D3	Balance 1 = Balance 0 - Principal 1

To generate the complete schedule copy A3.E3 to A4.E62  
 To get the last payment adjust B62 = E61+C62  
 To get totals apply  $\Sigma$  to B1.D62

Below are the first 3 months and the last 3 months of a complete amortization schedule together with the required totals.

	A	B	C	D	E
1	Pmt#	Payment	Interest	Principal	Balance
2	0				6000.00
3	1	116.00	30.00	86.00	5914.00
4	2	116.00	29.57	86.43	5827.57
5	3	116.00	29.14	86.86	5740.71
6	:	:	:	:	:
7	58	116.00	1.72	114.28	230.05
8	59	116.00	1.15	114.85	115.20
9	60	115.78	0.58	115.20	0.00
10		6959.78	959.78	6000.00	

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**EXAMPLE 5** A couple purchases a home and signs a mortgage contract for \$80 000 to be paid in equal monthly payments over 25 years with interest at  $j_2 = 10\frac{1}{2}\%$ . Find the monthly payment and make out a partial amortization schedule showing the distribution of the first payments as to interest and repayment of principal.

**Solution** Since the interest is compounded semi-annually and the payments are paid monthly we have a general annuity problem. First we calculate rate  $i$  per month equivalent to  $5\frac{1}{4}\%$  per half-year and store it in a memory of the calculator.

$$\begin{aligned}(1 + i)^{12} &= (1.0525)^2 \\(1 + i) &= (1.0525)^{1/6} \\i &= (1.0525)^{1/6} - 1 \\i &= .008564515\end{aligned}$$

The monthly payment  $R = \frac{80\,000}{a_{\overline{300}|i}} = \$742.67$ .

To make out the amortization schedule for the first 6 months, we follow the same method as in the previous two examples with  $i = .008564515$ .

Payment Number	Periodic Payment	Payment of Interest at $i$	Principal Repaid	Outstanding Balance
				\$80 000.00
1	742.67	685.16	57.51	79 942.49
2	742.67	684.67	58.00	79 884.49
3	742.67	684.17	58.50	79 825.99
4	742.67	683.67	59.00	79 766.99
5	742.67	683.17	59.50	79 707.49
6	742.67	682.66	60.01	79 647.48
Totals	4456.02	4103.50	352.52	

During the first 6 months only \$352.52 of the original \$80 000 debt is repaid. It should be noted that over 92% of the first six payments goes for interest and less than 8% for the reduction of the outstanding balance.

Again, the entries in the principal repaid column are in the ratio of  $(1 + i)$ . That is:

$$\frac{58.00}{57.51} \doteq \frac{58.50}{58.00} \doteq \dots \doteq \frac{60.01}{59.50} \doteq 1 + i$$

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**EXAMPLE 6** Prepare an Excel spreadsheet and show the first 6 months and the last 6 months of a complete amortization schedule for the mortgage loan of Example 5 above. Also show total payments, total interest and total principal paid.

**Solution:** We summarize the entries for line 0 and line 1 of an Excel spreadsheet.

CELL	ENTER
A2	0
E2	80,000.00
A3	=A2+1
B3	742.67
C3	= E2*((1 + 0.105/2)^(1/6)-1)
D3	=B3-C3
E3	=E2-D3

To generate the complete schedule copy A3.E3 to A4.E302

To get the last payment adjust B302 =E301+C302

To get the totals apply  $\Sigma$  to B1.D302

Below are the first 6 months and the last 6 months of a complete amortization schedule together with the required totals.

	A	B	C	D	E
1	Prnt#	Payment	Interest	Principal	Balance
2	0				80,000.00
3	1	742.67	685.16	57.51	79,952.49
4	2	742.67	684.67	58.00	79,884.49
5	3	742.67	684.17	58.50	79,825.99
6	4	742.67	683.67	59.00	79,766.99
7	5	742.67	683.17	59.50	79,707.49
8	6	742.67	682.66	60.01	79,647.47

	A	B	C	D	E
1	Pmt#	Payment	Interest	Principal	Balance
2	295	742.67	36.97	705.70	3,611.48
3	296	742.67	30.93	711.74	2,899.74
4	297	742.67	24.83	717.84	2,181.91
5	298	742.67	18.69	723.98	1,457.92
6	299	742.67	12.49	730.18	727.74
7	300	733.97	6.23	727.74	-
8		222,792.30	142,792.30	80,000.00	

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**EXAMPLE 7** General amortization schedule

Consider a loan of \$ $A$  to be repaid with level payments of \$ $R$  at the end of each period for  $n$  periods, at rate  $i$  per period. Show that in the  $k$ th line of the amortization schedule ( $1 \leq k \leq n$ )

a) Interest payment  $I_k = iRa_{\overline{n-k+1}|i} = R[1 - (1+i)^{-(n-k+1)}]$

b) Principal payment  $P_k = R(1+i)^{-(n-k+1)}$

c) Outstanding balance  $B_k = Ra_{\overline{n-k}|i}$

Verify that

d) The sum of the principal payments equals the original value of the loan.

e) The sum of the interest payments equals the total payments less the original value of the loan.

f) The principal payments are in the ratio  $(1+i)$ .

**Solution a** The outstanding balance after the  $(k-1)$ st payment is the discounted value of the remaining  $n - (k-1) = n - k + 1$  payments, that is

$$B_{k-1} = Ra_{\overline{n-k+1}|i}$$

Interest paid in the  $k$ -th payment is

$$I_k = iB_{k-1} = iRa_{\overline{n-k+1}|i} = iR \frac{1 - (1+i)^{-(n-k+1)}}{i} = R[1 - (1+i)^{-(n-k+1)}]$$

**Solution b** Principal repaid in the  $k$ -th payment is

$$P_k = R - I_k = R - R[1 - (1+i)^{-(n-k+1)}] = R(1+i)^{-(n-k+1)}$$

**Solution c** The outstanding balance after the  $k$ -th payment is the discounted value of the remaining  $n - k$  payment, that is

$$B_k = Ra_{\overline{n-k}|i}$$

**Solution d** The sum of the principal payments is

$$\sum_{k=1}^n P_k = \sum_{k=1}^n R(1+i)^{-(n-k+1)} = R[(1+i)^{-n} + \dots + (1+i)^{-1}] = Ra_{\overline{n}|i} = A$$

**Solution e** The sum of the interest payment is

$$\sum_{k=1}^n I_k = \sum_{k=1}^n R[1 - (1+i)^{-(n-k+1)}] = nR - Ra_{\overline{n}|i} = nR - A$$

**Solution f**  $\frac{P_{k+1}}{P_k} = \frac{R(1+i)^{-(n-(k+1)+1)}}{R(1+i)^{-(n-k+1)}} = (1+i)^{-(n+k)+(n-k+1)} = (1+i)$  ■

## Exercise 5.1

### Part A

1. A loan of \$5000 is to be amortized with equal quarterly payments over a period of 5 years at  $j_4 = 12\%$ . Find the concluding payment if the quarterly payment is rounded up to a) the cent; b) the dime.
2. A loan of \$20 000 is to be amortized with equal monthly payments over a 3-year period at  $j_{12} = 8\%$ . Find the concluding payment if the monthly payment is rounded up to a) the cent; b) the dime.
3. A \$5000 loan is to be amortized with 8 equal semi-annual payments. If interest is at  $j_2 = 14\%$ , find the semi-annual payment and construct an amortization schedule.
4. A loan of \$900 is to be amortized with 6 equal monthly payments at  $j_{12} = 12\%$ . Find the monthly payment and construct an amortization schedule.
5. A loan of \$1000 is to be repaid over 2 years with equal quarterly payments. Interest is at  $j_{12} = 6\%$ . Find the quarterly payment required and construct an amortization schedule to show the interest and principal portion of each payment.
6. A debt of \$50 000 with interest at  $j_4 = 8\%$  is to be amortized by payments of \$10 000 at the end of each quarter for as long as necessary. Create an amortization schedule.
7. A \$10 000 loan is to be repaid with semi-annual payments of \$2500 for as long as necessary. If interest is at  $j_{12} = 12\%$ , do a complete amortization schedule showing the distribution of each payment into principal and interest.
8. A debt of \$2000 will be repaid by monthly payments of \$500 for as long as necessary, the first payment to be made at the end of 6 months. If interest is at  $j_{12} = 9\%$ , find the size of the debt at the end of 5 months and make out the complete schedule starting at that time.
9. A couple buys some furniture for \$1500. They pay off the debt at  $j_{12} = 18\%$  by paying \$200 a month for as long as necessary. The first payment is at the end of 3 months. Do a complete amortization schedule for this loan showing the distribution of each payment into principal and interest.
10. A \$16 000 car is purchased by paying \$1000 down and then equal monthly payments for 3 years at  $j_{12} = 15\%$ . Find the size of the monthly payments and complete the first three lines of the amortization schedule.
11. A mobile home worth \$46 000 is purchased with a down payment of \$6000 and monthly payments for 15 years. If interest is  $j_2 = 10\%$ , find the monthly payment required and complete the first 6 lines of the amortization schedule.
12. A couple purchases a home worth \$256 000 by paying \$86 000 down and then taking out a mortgage at  $j_2 = 7\%$ . The mortgage will be amortized over 25 years with equal monthly payments. Find the monthly payment and do a partial amortization schedule showing the distribution of the first 6 payments as to interest and principal. How much of the principal is repaid during the first 6 months?

13. In September 1981, mortgage interest rates in Canada peaked at  $j_2 = 21\frac{1}{2}\%$ . Redo Question 12 using  $j_2 = 21\frac{1}{2}\%$ .

**Part B**

1. On a loan with  $j_{12} = 12\%$  and monthly payments, the amount of principal in the 6th payment is \$40.
  - a) Find the amount of principal in the 15th payment.
  - b) If there are 36 equal payments in all, find the amount of the loan.
2. A loan is being repaid over 10 years with equal annual payments. Interest is at  $j_1 = 10\%$ . If the amount of principal repaid in the third payment is \$100, find the amount of principal repaid in the 7th payment.
3. The ABC Bank develops a special scheme to help their customers pay their loans off quickly. Instead of making payments of  $\$X$  once a month, mortgage borrowers are asked to pay  $\frac{\$X}{4}$  once a week (52 times a year).  
The Gibsons are buying a house and need a \$180 000 mortgage. If  $j_2 = 8\%$ , determine
  - a) the monthly payment required to amortize the debt over 25 years;
  - b) the weekly payment  $\frac{\$X}{4}$  suggested in the scheme;
  - c) the number of weeks it will take to pay off the debt using the suggested scheme.
 Compare these results and comment.
4. A loan is being repaid by monthly instalments of \$100 at  $j_{12} = 18\%$ . If the loan balance after the fourth month is \$1200, find the original loan value.
5. A loan is being repaid with 20 annual instalments at  $j_1 = 15\%$ . In which instalment are the principal and interest portions most nearly equal to each other?
6. A loan is being repaid with 10 annual instalments. The principal portion of the seventh payment is \$110.25 and the interest portion is \$39.75. What annual effective rate of interest is being charged?
7. A loan at  $j_1 = 9\%$  is being repaid by monthly payments of \$750 each. The total principal repaid in the twelve monthly instalments of the 8th year is \$400. What is the total interest paid in the 12 instalments of the 10th year?
8. Below is part of a mortgage amortization schedule.

Payment	Interest	Principal	Balance
	\$243.07	\$31.68	
	242.81	31.94	

Determine

- a) the monthly payment;
  - b) the effective rate of interest per month;
  - c) the nominal rate of interest  $j_2$ , rounded to nearest  $\frac{1}{8}\%$  (Note: Use this rounded nominal rate from here on);
  - d) the outstanding balance just after the first payment shown above;
  - e) the remaining period of the mortgage if interest rates don't change.
9. On mortgages repaid by equal annual payments covering both principal and interest, a mortgage company pays as a commission to its agents 10% of the portion of each scheduled instalment, which represents interest. What is the total commission paid per \$1000 of original mortgage loan if it is repaid with  $n$  annual payments at rate  $i$  per year?

10. Fred buys a personal computer from a company whose head office is in the United States. The computer includes software designed to calculate mortgage amortization schedules. Fred has a \$75 000 mortgage with 20-year amortization at  $j_2 = 6\frac{1}{2}\%$ . According to his bank statement, his monthly payment is \$555.38. When Fred uses his software, he enters his original principal balance of \$75 000, the amortization period of 20 years and the nominal annual rate of interest of  $6\frac{1}{2}\%$ . The computer produces output that says the monthly payment should be \$559.18. Which answer is correct? Can you explain the error?
11. As part of the purchase of a home on January 1, 2000, you have just negotiated a mortgage in the amount of \$150 000. The amortization period for calculation of the level monthly payments (principal and interest) has been set at 25 years and the interest rate is 9% per annum compounded semi-annually.
- What level monthly payment is required, assuming the first one to be made at February 1, 2000?
  - It has been suggested that if you multiply the monthly payment [calculated in a)] by 12, divide by 52 then pay the resulting amount each week, with the first payment at January 8, 2000, then the amortization period will be shortened. If the lender agrees to this, at what time in the future will the mortgage be fully paid?
  - What will be the size of the smaller, final weekly payment, made 1 week after the last regular payment [as calculated for part b)]?
12. Part of an amortization schedule shows

Payment	Interest	Principal	Balance
	440.31	160.07	
	438.71	161.67	

Complete the next 2 lines of this schedule.

13. You lend a friend \$15 000 to be amortized by semi-annual payments for 8 years, with interest at  $j_2 = 9\%$ . You deposit each payment in an account paying  $j_{12} = 7\%$ . What annual effective rate of interest have you earned over the entire 8-year period?
14. A loan of \$10 000 is being repaid by semi-annual payments of \$1000 on account of principal. Interest on the outstanding balance at  $j_2$  is paid in addition to the principal repayments. The total of all payments is \$12 200. Find  $j_2$ .
15. A loan of \$A is to be repaid by 16 equal semi-annual instalments, including principal and interest, at rate  $i$  per half year. The principal in the first instalment (six months hence) is \$30.83. The principal in the last is \$100. Find the annual effective rate of interest.
16. A loan is to be repaid by 16 quarterly payments of \$50, \$100, \$150, ..., \$800, the first payment due three months after the loan is made. Interest is at a nominal annual rate of 8% compounded quarterly. Find the total amount of interest contained in the payments.
17. A loan of \$20 000 with interest at  $j_{12} = 6\%$  is amortized by equal monthly payments over 15 years. In which payment will the interest portion be less than the principal portion, for the first time?
18. You are choosing between two mortgages for \$160 000 with a 20-year amortization period. Both charge  $j_2 = 7\%$  and permit the mortgage to be paid off in less than 20 years. Mortgage A allows you to make weekly payments, with each payment being  $\frac{1}{4}$  of the normal monthly payment. Mortgage B allows you to make double the usual monthly payment every 6 months. Assuming that you will take advantage of the mortgage provisions, calculate the total interest charges over the life of each mortgage to determine which mortgage costs less.

19. In the United States, in an effort to advertise low rates of interest, but still achieve high rates of return, lenders sometimes charge *points*. Each point is a 1% discount from the face value of the loan. Suppose a home is being sold for \$220 000 and that the buyer pays \$60 000 down and gets a \$160 000 15-year mortgage at  $j_2 = 8\%$ . The lender charges 5 points, that is 5% of  $160\,000 = \$8000$ , so the loan is \$152 000, but \$160 000 is repaid. What is the true interest rate,  $j_2$  on the loan?

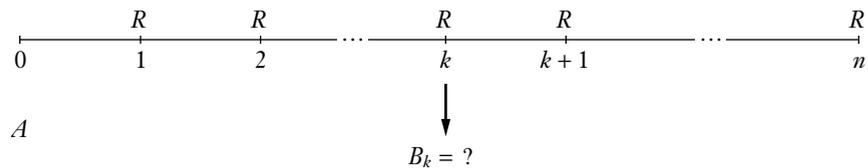
## Section 5.2

### Outstanding Balance

It is quite important to know the amount of principal remaining to be paid at a certain time. The borrower may want to pay off the outstanding balance of the debt in a lump sum, or the lender may wish to sell the contract.

One could find the outstanding balance by making out an amortization schedule. This becomes rather tedious without a spreadsheet when a large number of payments are involved. In this section we shall calculate the outstanding balance directly from an appropriate equation of value.

Let  $B_k$  denote the outstanding balance immediately after the  $k$ th payment has been made.



Two methods for finding  $B_k$  are available.

**The Retrospective Method:** This method uses the past history of the debt—the payments that have been made already. The outstanding balance  $B_k$  is calculated as the difference between the accumulated value of the debt and the accumulated value of the payments already made.

Thus,

$$B_k = A(1+i)^k - Rs\overline{k}|i \quad (13)$$

Equation (13) always gives the correct value of the outstanding balance  $B_k$  and can be used in all cases, even if the number of payments is not known or the last payment is an irregular one.

**The Prospective Method:** This method uses the future prospects of the debt—the payments yet to be made. The outstanding balance  $B_k$  is calculated as the discounted value of the  $(n-k)$  payments yet to be made.

If all the payments, including the last one are equal, we obtain

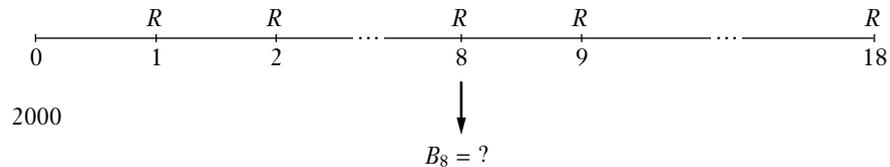
$$B_k = Ra\overline{n-k}|i \quad (14)$$

While equations (13) and (14) are algebraically equivalent (see Problem B9), equation (14) cannot be used when the concluding payment is an irregular one. However, it is useful if you don't know the original value of the loan. When the concluding payment is an irregular one, equation (14) is still applicable if suitably modified, but it is usually simpler to use the retrospective method than to discount the  $(n - k)$  payments yet to be made.

**EXAMPLE 1** A loan of \$2000 with interest at  $j_{12} = 12\%$  is to be amortized by equal payments at the end of each month over a period of 18 months. Find the outstanding balance at the end of 8 months.

**Solution** First we calculate the monthly payment  $R$ , given  $A = 2000$ ,  $n = 18$ ,  $i = .01$ ,

$$R = \frac{2000}{a_{\overline{18}|.01}} = \$121.97$$



**The retrospective method** We have  $A = 2000$ ,  $R = 121.97$ ,  $k = 8$ ,  $i = .01$  and calculate  $B_8$  using equation (13)

$$\begin{aligned} B_8 &= 2000(1.01)^8 - 121.97s_{\overline{8}|.01} \\ &= 2165.71 - 1010.60 = \$1155.11 \end{aligned}$$

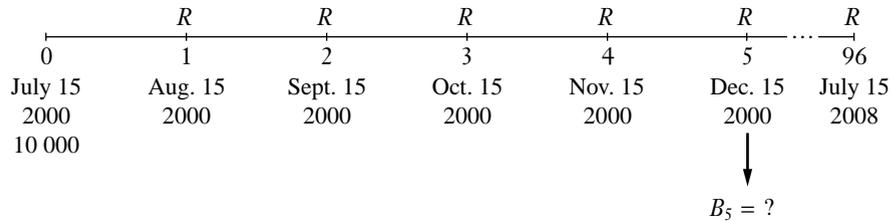
**The prospective method** We have  $R = 121.97$ ,  $n - k = 10$ ,  $i = .01$  and calculate  $B_8$  using equation (14)

$$B_8 = 121.97a_{\overline{10}|.01} = \$1155.21$$

The difference of 10 cents is due to rounding the monthly payment  $R$  up to the next cent. The concluding payment is, in fact, slightly smaller than the regular payment  $R = 121.97$ , so the value of  $B_8$  under the prospective method is not accurate. ■

**EXAMPLE 2** On July 15, 2000, a couple borrowed \$10 000 at  $j_{12} = 15\%$  to start a business. They plan to repay the debt in equal monthly payments over 8 years with the first payment on August 15, 2000. a) How much principal did they repay during 2000? b) How much interest can they claim as a tax deduction during 2000?

**Solution** We arrange our data on a time diagram below.



First we calculate the monthly payment  $R$ , given  $A = 10\,000$ ,  $n = 96$ ,  $i = .0125$

$$R = \frac{10\,000}{a_{\overline{96}|.0125}} = \$179.46$$

Then we calculate the outstanding balance  $B_5$  on December 15, 2000, after the 5th payment has been made. We have  $A = 10\,000$ ,  $R = 179.46$ ,  $k = 5$ ,  $i = .0125$ , and using equation (13) we calculate

$$\begin{aligned} B_5 &= 10\,000(1.0125)^5 - 179.46s_{\overline{5}|.0125} \\ &= 10\,640.82 - 920.01 \\ &= \$9720.81 \end{aligned}$$

The total reduction in principal in 2000 is the difference between the outstanding balance on December 15, 2000, after the 5th payment has been made, and the original debt of \$10 000. Thus, they repaid  $10\,000 - 9720.81 = \$279.19$  of principal during 2000.

To get the total interest paid in 2000, we subtract the amount they repaid on principal from the total of the 5 payments, i.e., total interest  $= 5 \times 179.46 - 279.19 = \$618.11$ . They can deduct \$618.11 as an expense on their 2000 income tax return. ■

This method of amortization is quite often used to pay off loans incurred in purchasing a property. In such cases, the outstanding balance is called the **seller's equity**. The amount of principal that has been paid already plus the down payment is called the **buyer's equity**, or **owner's equity**. At any point in time we have the following relation:

$$\text{Buyer's equity} + \text{Seller's equity} = \text{Original selling price}$$

The buyer's equity starts with the down payment and is gradually increased with each periodic payment by the part of the payment which is applied to reduce the outstanding balance. It should be noted that the buyer's equity, as defined above, does not make any allowance for increases or decreases in the value of the property.

**EXAMPLE 3** The Mancinis buy a cottage worth \$178 000 by paying \$38 000 down and the balance, with interest at  $j_2 = 11\%$ , in monthly instalments of \$2000 for as long as necessary. Find the Mancinis' equity at the end of 5 years.

**Solution** The monthly payments form a general annuity. First, we calculate a monthly rate of interest  $i$  equivalent to  $5\frac{1}{2}\%$  each half-year.

$$\begin{aligned}(1 + i)^{12} &= (1.055)^2 \\(1 + i) &= (1.055)^{\frac{1}{6}} \\i &= (1.055)^{\frac{1}{6}} - 1 \\i &= .008963394\end{aligned}$$

Using the retrospective method, we calculate the seller's equity as the outstanding balance at the end of 60 months.

$$\begin{aligned}B_{60} &= 140\,000(1 + i)^{60} - 2000\overline{s}_{\overline{60}|i} \\&= 239\,140.20 - 158\,008.10 = \$81\,132.10\end{aligned}$$

The Mancinis' equity is then

$$\text{Buyer's equity} = 178\,000 - 81\,132.10 = \$96\,867.90.$$

■

## Exercise 5.2

### Part A

- To pay off the purchase of a car, Chantal got a \$15 000, 3-year bank loan at  $j_{12} = 12\%$ . She makes monthly payments. How much does she still owe on the loan at the end of two years (24 payments)? Use both the retrospective and prospective methods.
- A debt of \$10 000 will be amortized by payments at the end of each quarter of a year for 10 years. Interest is at  $j_4 = 10\%$ . Find the outstanding balance at the end of 6 years.
- On July 1, 2000, Brian borrowed \$30 000 to be repaid with monthly payments (first payment August 1, 2000) over 3 years at  $j_{12} = 8\%$ . How much principal did he repay in 2000? How much interest?
- A couple buys a house worth \$156 000 by paying \$46 000 down and taking out a mortgage for \$110 000. The mortgage is at  $j_2 = 11\%$  and will be repaid over 25 years with monthly payments. How much of the principal does the couple pay off in the first year?
- On May 1, 2000, the Morins borrow \$4000 to be repaid with monthly payments over 3 years at  $j_{12} = 9\%$ . The 12 payments made during 2001 will reduce the principal by how much? What was the total interest paid in 2001?
- To pay off the purchase of home furnishings, a couple takes out a bank loan of \$2000 to be repaid with monthly payments over 2 years at  $j_{12} = 15\%$ . What is the outstanding debt just after the 10th payment? What is the principal portion of the 11th payment?
- A couple buys a piece of land worth \$200 000 by paying \$50 000 down and then taking a loan out for \$150 000. The loan will be repaid with quarterly payments over 15 years and is at  $j_4 = 12\%$ . Find the couple's equity at the end of 8 years.
- A family buys a house worth \$326 000. They pay \$110 000 down and then take out a 5-year mortgage for the balance at  $j_2 = 6\frac{1}{2}\%$  to be amortized over 20 years. Payments will be made monthly. Find the outstanding balance at the end of 5 years and the owners' equity at that time.

9. Land worth \$80 000 is purchased by a down payment of \$12 000 and the balance in equal monthly instalments for 15 years. If interest is at  $j_{12} = 9\%$ , find the buyer's and seller's equity in the land at the end of 9 years.
10. A loan of \$10 000 is being repaid by instalments of \$200 at the end of each month for as long as necessary. If interest is at  $j_4 = 8\%$ , find the outstanding balance at the end of 1 year.
11. A debt is being amortized at an annual effective rate of interest of 15% by payments of \$500 made at the end of each year for 11 years. Find the outstanding balance just after the 7th payment.
12. A loan is to be amortized by semi-annual payments of \$802, which includes principal and interest at 8% compounded semi-annually. What is the original amount of the loan if the outstanding balance is reduced to \$17 630 at the end of 3 years?
13. A loan is being repaid with semi-annual instalments of \$1000 for 10 years at 10% per annum convertible semi-annually. Find the amount of principal in the 6th instalment.
14. Jones purchased a cottage, paying \$10 000 down and agreeing to pay \$500 at the end of every month for the next ten years. The rate of interest is  $j_2 = 12\%$ . Jones discharges the remaining indebtedness without penalty by making a single payment at the end of 5 years. Find the extra amount that Jones pays in addition to the regular payment then due.

### Part B

1. With mortgage rates at  $j_2 = 11\%$ , the XYZ Trust Company makes a special offer to its customers. It will lend mortgage money and determine the monthly payment as if  $j_2 = 9\%$ . The mortgage will be carried at  $j_2 = 11\%$  and any deficiency that results will be added to the outstanding balance. If the Moras are taking out a \$100 000 mortgage to be repaid over 25 years under this scheme, what will their outstanding balance be at the end of 5 years?
2. The Hwangs can buy a home for \$190 000. To do so would require taking out a \$125 000 mortgage from a bank at  $j_2 = 11\%$ . The loan will be repaid over 25 years with the rate of interest fixed for 5 years.  
The seller of the home is willing to give the Hwangs a mortgage at  $j_2 = 10\%$ . The monthly payment will be determined using a 25-year repayment schedule. The seller will guarantee the rate of interest for 5 years at which time the Hwangs will have to pay off the seller and get a mortgage from a bank. If the Hwangs accept this offer, the seller wants \$195 000 for the house, forcing the Hwangs to borrow \$130 000. If the Hwangs can earn  $j_{12} = 8\%$  on their money, what should they do?
3. The Smiths buy a home and take out a \$160 000 mortgage on which the interest rate is allowed to float freely. At the time the mortgage is issued, interest rates are  $j_2 = 10\%$  and the Smiths choose a 25-year amortization schedule. Six months into the mortgage, interest rates rise to  $j_2 = 12\%$ . Three years into the mortgage (after 36 payments) interest rates drop to  $j_2 = 11\%$  and four years into the mortgage, interest rates drop to  $j_2 = 9\frac{1}{2}\%$ . Find the outstanding balance of the mortgage after 5 years. (The monthly payment is set at issue and does not change.)
4. A young couple buys a house and assumes a \$90 000 mortgage to be amortized over 25 years. The interest rate is guaranteed at  $j_2 = 8\%$ . The mortgage allows the couple to make extra payments against the outstanding principal each month. By saving carefully the couple manages to pay off an extra \$100 each month. Because of these extra payments, how long will it take to pay off the mortgage and what will be the size of the final smaller monthly payment?

5. A loan is made on January 1, 1985, and is to be repaid by 25 level annual instalments. These instalments are in the amount of \$3000 each and are payable on December 31 of the years 1985 through 2009. However, just after the December 31, 1989, instalment has been paid, it is agreed that, instead of continuing the annual instalments on the basis just described, henceforth instalments will be payable quarterly with the first such quarterly instalment being payable on March 31, 1990, and the last one on December 31, 2009. Interest is at an annual effective rate of 10%. By changing from the old repayment schedule to the new one, the borrower will reduce the total amount of payments made over the 25-year period. Find the amount of this reduction.
6. The ABC Trust Company issues loans where the monthly payments are determined by the rate of interest that prevails on the day the loan is made. After that, the rate of interest varies according to market forces but the monthly payments do not change in dollar size. Instead, the length of time to full repayment is either lengthened (if interest rates rise) or shortened (if interest rates fall).  
Medhaf takes out a 10-year, \$20 000 loan at  $j_2 = 8\%$ . After exactly 2 years (24 payments) interest rates change. Find the duration of the remaining loan and the final smaller payment if the new interest rate is a)  $j_2 = 9\%$ ; b)  $j_2 = 7\%$ .
7. Big Corporation built a new plant in 1997 at a cost of \$1 700 000. It paid \$200 000 cash and assumed a mortgage for \$1 500 000 to be repaid over 10 years by equal semi-annual payments due each June 30 and December 31, the first payment being due on December 31, 1997. The mortgage interest rate is 11% per annum compounded semi-annually and the original date of the loan was July 1, 1997.
  - a) What will be the total of the payments made in 1999 on this mortgage?
  - b) Mortgage interest paid in any year (for this mortgage) is an income tax deduction for that year. What will be the interest deduction on Big Corporation's 1999 tax form?
  - c) Suppose the plant is sold on January 1, 2001. The buyer pays \$650 000 cash and assumes the outstanding mortgage. What is Big Corporation's capital gain (amount realized less original price) on the investment in the building?
8. Given a loan  $L$  to be repaid at rate  $i$  per period with equal payments  $R$  at the end of each period for  $n$  periods, give the Retrospective and Prospective expressions for the outstanding balance of the loan at time  $k$  (after the  $k$ th payment is made) and prove that the two expressions are equal.
9. A 5-year loan is being repaid with level monthly instalments at the end of each month, beginning with January 2000 and continuing through December 2004. A 12% nominal annual interest rate compounded monthly was used to determine the amount of each monthly instalment. On which date will the outstanding balance of this loan first fall below one-half of the original amount of the loan?
10. A debt is amortized at  $j_4 = 10\%$  by payments of \$300 per quarter. If the outstanding principal is \$2853.17 just after the  $k$ th payment, what was it just after the  $(k - 1)$ st payment?
11. A loan of \$3000 is to be repaid by annual payments of \$400 per annum for the first 5 years and payments of \$450 per year thereafter for as long as necessary. Find the total number of payments and the amount of the smaller final payment made one year after the last regular payment. Assume an annual effective rate of 7%.

12. Five years ago, Justin deposited \$1000 into a fund out of which he draws \$100 at the end of each year. The fund guarantees interest at 5% on the principal on deposit during the year. If the fund actually earns interest at a rate in excess of 5%, the excess interest earned during the year is paid to Justin at the end of the year in addition to the regular \$100 payment. The fund has been earning 8% each year for the past 5 years. What is the total payment Justin now receives?
13. An advertisement by the Royal Trust said,  
Our new Double-Up mortgage can be paid off faster and that dramatically reduces the interest you pay. You can double your payment any or every month, with no penalty. Then your payment reverts automatically to its normal amount the next month. What this means to you is simple. You will pay your mortgage off sooner. And that's good news because you can save thousands of dollars in interest as a result.  
Consider a \$120 000 mortgage at 8% per annum compounded semi-annually, amortized over 25 years with no anniversary pre payments.
- Find the required monthly payment for the above mortgage.
  - What is the total amount of interest paid over the full amortization period (assuming  $j_2 = 8\%$ )?
  - Suppose that payments were “Doubled-Up,” according to the advertisement, every 6th and 12th month
    - How many years and months would be required to pay off the mortgage?
    - What would be the total amount of interest paid over the full amortization period?
    - How much of the loan would still be outstanding at the end of 3 years, just after the Doubled-Up payment then due?
    - How much principal is repaid in the payment due in 37 months?

### Section 5.3

## Refinancing a Loan—The Amortization Method

It is common to want to renegotiate a long-term loan after it has been partially paid off.

If the loan is refinanced at a lower rate, the discounted value of the savings due to lower interest charges must be compared with the cost of refinancing to decide whether the refinancing would be a profitable one.

**EXAMPLE 1** Mr. Bouchard buys \$5000 worth of home furnishings from the ABC Furniture Mart. He pays \$500 down and agrees to pay the balance with monthly payments over 5 years at  $j_{12} = 18\%$ . The contract he signs stipulates that if he pays off the contract early there is a penalty equal to three months' payments. After two years (24 payments) Mr. Bouchard realizes that he could borrow the money from the bank at  $j_{12} = 12\%$ . He realizes that to do so means he will have to pay the three-month penalty on the ABC Furniture Mart contract. Should he refinance?

**Solution** First, find the monthly payments required under the original contract.

$$R_1 = \frac{4500}{a_{\overline{60}|0.015}} = \$114.28$$

Now, find the outstanding balance  $B_{24}$  on the original contract at the end of two years.

$$B_{24} = 4500(1.015)^{24} - 114.28s_{\overline{24}|0.015} = \$3160.52$$

If Mr. Bouchard repays the loan he will also pay a penalty equal to  $3 \times R = \$342.84$ . Therefore, the amount he must borrow from the bank is

$$3160.52 + 342.84 = \$3503.36$$

Thus, the new monthly payments on the bank loan are

$$R_2 = \frac{3503.62}{a_{\overline{36}|0.01}} = \$116.37$$

Therefore, he should not refinance since \$116.37 is larger than \$114.28. ■

The largest single loan the average Canadian is likely to make will be the mortgage on one's home. At one time, interest rates on mortgages were guaranteed for the life of the repayment schedule, which could be as long as 25 or 30 years. Now, the longest period of guaranteed interest rates available is usually 5 years, although homeowners may choose mortgages where the interest rate is adjusted every three years, every year or even daily to current interest rates.

A homeowner who wishes to repay the mortgage in full before the defined renegotiation date will have to pay a penalty. In some cases, this penalty is defined as three months of interest on the amount prepaid. In other cases the penalty varies according to market conditions and can be determined by the lender at the time of prepayment (the formula that the lender must use may be defined in the mortgage contract, however). This situation is illustrated in Example 3.

With the high interest rates of the late 1980s and the drop in rates in the early 1990s, many new and different mortgages are being offered to the prospective borrower. It is important that students are capable of analyzing these contracts fully. Several examples are illustrated in the exercises contained in this chapter.

**EXAMPLE 2** A couple purchased a home and signed a mortgage contract for \$180 000 to be paid with monthly payments calculated over a 25-year period at  $j_2 = 10\%$ . The interest rate is guaranteed for 5 years. After 5 years, they renegotiate the interest rate and refinance the loan at  $j_2 = 6\frac{1}{2}\%$ . There is no penalty if a mortgage is refinanced at the end of an interest rate guarantee period. Find

- the monthly payment for the initial 5-year period;
- the new monthly payments after 5 years;
- the accumulated value of the savings for the second 5-year period (at the end of that period) at  $j_{12} = 4\frac{1}{2}\%$ ; and
- the outstanding balance at the end of 10 years.

**Solution a** First we calculate a monthly rate of interest  $i$  equivalent to  $j_2 = 10\%$

$$\begin{aligned}(1 + i)^{12} &= (1.05)^2 \\ (1 + i) &= (1.05)^{\frac{1}{6}} \\ i &= (1.05)^{\frac{1}{6}} - 1 \\ i &= .008164846\end{aligned}$$

Now  $A = 180\,000$ ,  $n = 300$ ,  $i = .008164846$  and we calculate the monthly payment  $R_1$  for the initial 5-year period.

$$R_1 = \frac{180\,000}{a_{300|i}} = \$1610.08$$

**Solution b** First we calculate the outstanding balance of the loan after 5 years.

$$\begin{aligned}180\,000(1 + i)^{60} - 1610.08s_{\overline{60}|i} \\ = 293\,201.03 - 124\,015.89 = \$169\,185.12\end{aligned}$$

This outstanding balance is refinanced at  $j_2 = 6\frac{1}{2}\%$  over a 20-year period. First, find the rate  $i$  per month equivalent to  $j_2 = 6\frac{1}{2}\%$ .

$$\begin{aligned}(1 + i)^{12} &= (1.0325)^2 \\ (1 + i) &= (1.0325)^{\frac{1}{6}} \\ i &= (1.0325)^{\frac{1}{6}} - 1 \\ i &= .00534474\end{aligned}$$

Now  $A = 169\,185.12$ ,  $n = 240$ ,  $i = .00534474$  and we calculate the new monthly payment  $R_2$ .

$$R_2 = \frac{169\,185.12}{a_{240|i}} = \$1252.82$$

**Solution c** The monthly savings after the loan is renegotiated after 5 years at  $j_2 = 6\frac{1}{2}\%$  is

$$1610.08 - 1252.82 = \$357.26$$

If these savings are deposited in an account paying  $j_{12} = 4\frac{1}{2}\%$ , the accumulated value of the savings at the end of the 5-year period is:

$$357.26s_{\overline{60}|.00375} = \$23\,988.42$$

**Solution d** The outstanding balance of the loan at the end of 10 years at  $i = .00534474$  is

$$\begin{aligned}169\,185.12(1 + i)^{60} - 1252.82s_{\overline{60}|i} \\ = 232\,950.03 - 88\,344.94 = \$144\,605.09\end{aligned}$$

The outstanding balance at the end of 10 years is \$144 605.09. That means that only  $180\,000 - 144\,605.09 = \$35\,394.91$  of the loan was repaid during the first 10 years (they still owe 80.3% of the original \$180 000 loan). This is despite the fact that payments in the first 10 years totalled  $(60 \times 1610.08 + 60 \times 1252.82)$  or \$171 774.

■

**EXAMPLE 3** The Knapps buy a house and borrow \$145 000 from the ABC Insurance Company. The loan is to be repaid with monthly payments over 30 years at  $j_2 = 9\%$ . The interest rate is guaranteed for 5 years. After exactly two years of making payments, the Knapps see that interest rates have dropped to  $j_2 = 7\%$  in the market place. They ask to be allowed to repay the loan in full so they can refinance. The Insurance Company agrees to renegotiate but sets a penalty exactly equal to the money the company will lose over the next 3 years. Find the value of the penalty.

**Solution** First, find the monthly payments  $R$  required on the original loan at  $j_2 = 9\%$ . Find  $i$  such that

$$\begin{aligned}(1 + i)^{12} &= (1.045)^2 \\ (1 + i) &= (1.045)^{\frac{1}{6}} \\ i &= (1.045)^{\frac{1}{6}} - 1 \\ i &= .007363123\end{aligned}$$

Now  $A = 145\,000$ ,  $n = 360$ ,  $i = .007363123$ , and

$$R = \frac{145\,000}{a_{\overline{360}|i}} = \$1149.61$$

Next we find the outstanding balance after 2 years

$$\begin{aligned}&= 145\,000(1 + i)^{24} - 1149.61s_{\overline{24}|i} \\ &= 172\,915.20 - 30\,058.08 = \$142\,857.12\end{aligned}$$

Also, we find the outstanding balance after 5 years if the loan is not renegotiated.

$$\begin{aligned}&145\,000(1 + i)^{60} - 1149.61s_{\overline{60}|i} \\ &= 225\,180.57 - 86\,335.54 = \$138\,845.03\end{aligned}$$

If the insurance company renegotiates, it will receive \$142 857.12 plus the penalty  $\$X$  now. If they do not renegotiate, they will receive \$1149.61 a month for 3 years plus \$138 845.03 at the end of 3 years. These two options should be equivalent at the current interest rate  $j_2 = 7\%$ .

	+ $X$											
Option 1:	142 857.12											
	0	1	2	...	35	36						
Option 2:		1149.61	1149.61		1149.61	1149.61	1149.61					
							+138 845.03					

Find the monthly rate  $i$  equivalent to  $j_2 = 7\%$ .

$$\begin{aligned}(1 + i)^{12} &= (1.035)^2 \\ (1 + i) &= (1.035)^{\frac{1}{6}} \\ i &= (1.035)^{\frac{1}{6}} - 1 \\ i &= .00575004\end{aligned}$$

Using 0 as the focal date we solve an equation of value for  $X$ .

$$\begin{aligned} X + 142\,857.12 &= 1149.61a_{\overline{36}|i} + 138\,845.03(1+i)^{-36} \\ X + 142\,857.12 &= 37\,286.97 + 112\,950.52 \\ X &= \$7380.37 \end{aligned}$$

Thus the penalty at the end of 2 years is \$7380.37. If the penalty had been three times the monthly interest on the outstanding balance at the end of 2 years, it would have been

$$3 \times 142\,857.12 \times .007363123 = \$3155.62$$

■

**EXAMPLE 4** The Wongs borrow \$10 000 from a bank to buy some home furnishings. Interest is  $j_{12} = 9\%$  and the term of the loan is 3 years. After making 15 monthly payments, the Wongs miss the next two monthly payments. The bank forces them to renegotiate the loan at a new, higher interest rate  $j_{12} = 10\frac{1}{2}\%$ . Determine

- the original monthly payments  $R_1$ ;
- the outstanding balance of the loan at the time the 17th monthly payment would normally have been made.
- the new monthly payments  $R_2$  if the original length of the loan term is not extended.

**Solution a** Using  $A = 10\,000$ ,  $n = 36$ ,  $i = \frac{.09}{12} = .0075$  we calculate the original monthly payment

$$R_1 = \frac{10\,000}{a_{\overline{36}|i}} = \$318.00$$

**Solution b** The first 15 monthly payments of \$318 were made. Therefore at the end of 15 months the outstanding balance was

$$\begin{aligned} &10\,000(1+i)^{15} - 318s_{\overline{15}|i} \\ &= 11\,186.03 - 5028.75 = \$6157.28 \end{aligned}$$

Then no payments were made for 2 months. Therefore the outstanding balance at the end of 17 months is

$$\$6157.28(1+i)^2 = \$6249.99$$

**Solution c** We have  $A = 6249.99$ ,  $n = 19$ ,  $i = \frac{.105}{12} = .00875$  and we calculate the new monthly payment

$$R_2 = \frac{6249.99}{a_{\overline{19}|i}} = \$358.49$$

■

### Exercise 5.3

#### Part A

1. A borrower is repaying a \$5000 loan at  $j_{12} = 15\%$  with monthly payments over 3 years. Just after the 12th payment (at the end of 1 year) he has the balance refinanced at  $j_{12} = 12\%$ . If the number of payments remains unchanged, what will be the new monthly payment and what will be the monthly savings in interest?
2. A 5-year, \$6000 loan is being amortized with monthly payments at  $j_{12} = 18\%$ . Just after making the 30th payment, the borrower has the balance refinanced at  $j_{12} = 12\%$  with the term of the loan to remain unchanged. What will be the monthly savings in interest?
3. A borrower has a \$5000 loan with the “Easy-Credit” Finance Company. The loan is to be repaid over 4 years at  $j_{12} = 24\%$ . The contract stipulates an early repayment penalty equal to 3 months’ payments. Just after the 20th payment, the borrower determines that his local bank would lend him money at  $j_{12} = 16\%$ . Should he refinance?
4. The Jones family buys a fridge and stove totalling \$1400 from their local appliance store. They agree to pay off the total amount with monthly payments over 3 years at  $j_{12} = 15\%$ . If they wish to pay off the contract early they will experience a penalty equal to 3 months’ interest on the effective outstanding balance. After 12 payments they see that interest rates at their local bank are  $j_{12} = 11\%$ . Should they refinance?
5. Consider a couple who bought a house in Canada in 1976. Assume they needed a \$60 000 mortgage, which was to be repaid with monthly payments over 25 years. In 1976, interest rates were  $j_2 = 10\frac{1}{2}\%$ . What was their monthly payment? In 1981 (on the 5th anniversary of their mortgage) their mortgage was renegotiated to reflect current market rates. The repayment schedule was to cover the remaining 20 years and interest rates were now  $j_2 = 22\%$ . What was the new monthly payment? What effect might this have on homeowners?
6. The Steins buy a house and take out an \$85 000 mortgage. The mortgage is amortized over 25 years at  $j_2 = 9\%$ . After  $3\frac{1}{2}$  years, the Steins sell their house and the buyer wants to set up a new mortgage better tailored to his needs. The Steins find out that in addition to repaying the principal balance on their mortgage, they must pay a penalty equal to three months’ interest on the outstanding balance. What total amount must they repay?
7. A couple borrows \$15 000 to be repaid with monthly payments over 48 months at  $j_4 = 10\%$ . They make the first 14 payments and miss the next three. What new monthly payments over 31 months would repay the loan on schedule?
8. A loan of \$20 000 is to be repaid in 13 equal annual payments, each payable at year-end. Interest is at  $j_1 = 8\%$ . Because the borrower is having financial difficulties, the lender agrees that the borrower may skip the 5th and 6th payments. Immediately after the 6th payment would have been paid, the loan is renegotiated to yield  $j_1 = 10\%$  for the remaining 7 years. Find the new level annual payment for each of the remaining 7 years.
9. A loan of \$50 000 was being repaid by monthly level instalments over 20 years at  $j_{12} = 9\%$  interest. Now, when 10 years of the repayment period are still to run, it is proposed to increase the interest rate to  $j_{12} = 10\frac{1}{2}\%$ . What should the new level payment be so as to liquidate the loan on its original due date?

10. The Mosers buy a camper trailer and take out a \$15 000 loan. The loan is amortized over 10 years with monthly payments at  $j_2 = 18\%$ .
- Find the monthly payment needed to amortize this loan.
  - Find the amount of interest paid by the first 36 payments.
  - After 3 years (36 payments) they could refinance their loan at  $j_2 = 16\%$  provided they pay a penalty equal to three months' interest on the outstanding balance. Should they refinance? (Show the difference in their monthly payments.)

### Part B

- A couple buys a home and signs a mortgage contract for \$120 000 to be paid with monthly payments over a 25-year period at  $j_2 = 10\frac{1}{2}\%$ . After 5 years, they renegotiate the interest rate and refinance the loan at  $j_2 = 7\%$ ; find
  - the monthly payment for the initial 5-year period;
  - the new monthly payment after 5 years;
  - the accumulated value of the savings for the second 5-year period at  $j_{12} = 3\%$  valued at the end of the second 5-year period;
  - the outstanding balance at the end of 10 years.
- Mrs. McDonald is repaying a debt with monthly payments of \$100 over 5 years. Interest is at  $j_{12} = 12\%$ . At the end of the second year she makes an extra payment of \$350. She then shortens her payment period by 1 year and renegotiates the loan without penalty and without an interest change. What are her new monthly payments over the remaining two years?
- Mr. Fisher is repaying a loan at  $j_{12} = 15\%$  with monthly payments of \$1500 over 3 years. Due to temporary unemployment, Mr. Fisher missed making the 13th through the 18th payments inclusive. Find the value of the revised monthly payments needed starting in the 19th month if the loan is still to be repaid at  $j_{12} = 15\%$  by the end of the original 3 years.
- Mrs. Metcalf purchased property several years ago, (but she cannot remember exactly when). She has a statement in her possession, from the mortgage company, which shows that the outstanding loan amount on January 1, 2000, is \$28 416.60 and that the current interest rate is 11% per annum compounded semi-annually. It further shows that monthly payments are \$442.65 (Principal and Interest only).
 

Assuming continuation of the interest rate until maturity and that monthly payments are due on the 1st of every month

  - What is the final maturity date of the mortgage?
  - What will be the amount of the final payment?
  - Show the amortization schedule entries for January 1 and February 1, 2000.
  - Calculate the new monthly payment required (effective after December 31, 1999) if the interest rate is changed to 8% per annum compounded semi-annually and the other terms of the mortgage remain the same.
  - Calculate the new monthly payment if Mrs. Metcalf decides that she would like to have the mortgage completely repaid by December 31, 2009 (i.e., last payment at December 1, 2009). Assume the 11% per annum compounded semi-annually interest rate, and that new payments will start on February 1, 2000.
- A loan effective January 1, 1995, is being amortized by equal monthly instalments over 5 years using interest at a nominal annual rate of 12% compounded monthly. The first such instalment was due February 1, 1995, and the last such instalment was to be due January 1, 2000. Immediately after the 24th instalment was made on January 1, 1997, a new level monthly

instalment is determined (using the same rate of interest) in order to shorten the total amortization period to  $3\frac{1}{2}$  years, so the final instalment will fall due on July 1, 1998. Find the ratio of the new monthly instalment to the original monthly instalment.

6. A \$150 000 mortgage is to be amortized by monthly payments for 25 years. Interest is  $j_2 = 9\frac{1}{2}\%$  for 5 years and could change at that time. No penalty is charged for full or partial payment of the mortgage after 5 years.
  - a) Calculate the regular monthly payment and the reduced final mortgage payment assuming the  $9\frac{1}{2}\%$  rate continues for the entire 25 years.
  - b) What is the outstanding balance after
    - i) 2 years; ii) 5 years?
  - c) During the 5-year period, there is a penalty of 6 months' interest on any principal repaid early. After 2 years, interest rates fall to  $j_2 = 8\%$  for 3-year mortgages. Calculate the new monthly payment if the loan is refinanced and set up to have the same outstanding balance at the end of the initial 5-year period. Would it pay to refinance? (Exclude consideration of any possible refinancing costs other than the penalty provided in the question.)
7. A \$200 000 mortgage is taken out at  $j_2 = 6\frac{1}{2}\%$ , to be amortized over 25 years by monthly payments. Assume that the  $6\frac{1}{2}\%$  rate continues for the entire life of the mortgage.
  - a) Find the regular monthly payment and the reduced final payment.
  - b) Find the accumulated value of all the interest payments on the mortgage, at the time of the final payment.
  - c) Find the outstanding principal after 5 years.
  - d) If an extra \$5000 is paid off in 5 years (no penalty) and the monthly payments continue as before, how much less will have to be paid over the life of the mortgage?
8. Two years ago the Tongs took out a \$175 000 mortgage that was to be amortized over a 25-year period with monthly payments. The initial interest rate was set at  $j_2 = 11\frac{1}{4}\%$  and guaranteed for 5 years. After exactly two years of payments, the Tongs see that mortgage interest rates for a 3-year term are  $j_2 = 7\frac{1}{2}\%$ . They ask the bank to let them pay off the old mortgage in full and take out a new mortgage with a 23-year amortization schedule with  $j_2 = 7\frac{1}{2}\%$  guaranteed for 3 years. The bank replies that the Tongs must pay a penalty equal to the total dollar difference in the interest payments over the next three years. The mortgage allows the Tongs to make a 10% lump-sum principal repayment at any time (i.e., 10% of the remaining outstanding balance). The Tongs argue that they should be allowed to make this 10% lump-sum payment first and then determine the interest penalty. How many dollars will they save with respect to the interest penalty if they are allowed to make the 10% lump-sum repayment?
9. Mr. Adams has just moved to Waterloo. He has been told by his employer that he will be transferred out of Waterloo again in exactly three years. Mr. Adams is going to buy a house and requires a \$150 000 mortgage. He has his choice of two mortgages with monthly payments and 25-year amortization period.

Mortgage A is at  $j_2 = 10\%$ . This mortgage stipulates, however, that if you pay off the mortgage any time before the fifth anniversary, you will have to pay a penalty equal to three month's interest on the outstanding balance at the time of repayment.

Mortgage B is at  $j_2 = 10\frac{1}{2}\%$  but can be paid off at any time without penalty. Given that Mr. Adams will have to repay the mortgage in three years and that he can save money at  $j_1 = 6\%$ , which mortgage should he choose?

## Section 5.4

**Refinancing a Loan – The Sum-of-Digits Method**

A few Canadian lending institutions, when determining the outstanding principal on a consumer loan, do not use the amortization method outlined in the previous two sections, but rather they use an approximation to the amortization method called the **Sum-of-Digits Method** or the **Rule of 78**.

We know that under the amortization method, each payment made on a loan is used partly to pay off interest owing and partly to pay off principal owing. We also know that the interest portion of each payment is largest at the early instalments and gets progressively smaller over time. Thus, if a consumer borrowed \$1000 at  $j_{12} = 12\%$  and repaid the balance over 12 months, we would get the following amortization schedule:

Payment Number	Periodic Payment	Payment of Interest at 1%	Principal Repaid	Outstanding Balance
				\$1000.00
1	88.85	10.00	78.85	921.15
2	88.85	9.21	79.64	841.51
3	88.85	8.42	80.43	761.08
4	88.85	7.61	81.24	679.84
5	88.85	6.80	82.05	597.79
6	88.85	5.98	82.87	514.92
7	88.85	5.15	83.70	431.22
8	88.85	4.31	84.54	346.68
9	88.85	3.47	85.38	261.30
10	88.85	2.61	86.24	175.06
11	88.85	1.75	87.10	87.96
12	88.84	0.88	87.96	0
Totals	1066.20	66.20	1000.00	

Under the sum-of-digits approximation, we find the interest portion of each payment by taking a defined proportion of the total interest paid—that is, \$66.20. The proportion used is found as follows.

The loan is paid off over 12 months. If we sum the digits from 1 to 12 we get 78. We now say that the amount of interest assigned to the first payment is  $\frac{12}{78} \times 66.20 = \$10.18$ . The amount of interest assigned to the second payment is  $\frac{11}{78} \times 66.20 = \$9.34$ . This process continues until the amount of interest assigned to the last payment is  $\frac{1}{78} \times 66.20 = \$0.85$ . In general, the interest portion of the  $k$ th payment,  $I_k$ , is given by:

$$I_k = \frac{n - k + 1}{s} I$$

where  $n$  = number of payments  
 $s$  = sum of digits from 1 to  $n = \frac{n(n+1)}{2}$   
 $I$  = total interest

Notice that the total of the interest column is still \$66.20 since  $(\frac{12}{78} + \frac{11}{78} + \dots + \frac{1}{78}) = \frac{78}{78}$ . Since many consumer loans are paid off over 12 months, the alternative name, the Rule of 78, is sometimes used.

If we apply the sum-of-digits method to the loan described above, we will get the repayment schedule shown below.

Payment Number	Periodic Payment	Payment of Interest	Principal Repaid	Outstanding Balance
				\$1000.00
1	88.85	10.18	78.67	921.33
2	88.85	9.34	79.51	841.82
3	88.85	8.49	80.36	761.46
4	88.85	7.64	81.21	680.25
5	88.85	6.79	82.06	598.19
6	88.85	5.94	82.91	515.28
7	88.85	5.09	83.76	431.52
8	88.85	4.24	84.61	346.91
9	88.85	3.39	85.46	261.45
10	88.85	2.55	86.30	175.15
11	88.85	1.70	87.15	88.00
12	88.85	0.85	88.00	0
Totals	1066.20	66.20	1000.00	

One important point to notice here is that while the error is small, the use of the sum-of-digits approximation leads to an outstanding balance that is always larger than that obtained by using the amortization method. Thus, if you want to refinance your loan to repay the balance early, there is a penalty attached because of the use of the sum-of-digits approximation.

In fact, in the above illustration we can see that if the consumer paid his loan off in full after one month the lending institution would have realized an interest return of \$10.18 on \$1000 over one month, or  $i = 1.018\%$ . This corresponds to  $j_{12} = 12.216\%$  as opposed to  $j_{12} = 12\%$ .

As the term of the loan lengthens and as the interest rate rises, the penalty involved in the use of the sum-of-digits method increases as Example 1 will show.

**EXAMPLE 1** Consider a \$6000 loan that is to be repaid over 5 years at  $j_{12} = 6\%$ . Show the first three lines of the repayment schedule using: a) the amortization method; b) the sum-of-digits method.

**Solution** First we calculate the monthly payment  $R$  required

$$R = \frac{6000}{a_{\overline{60}|0.005}} = \$116.00$$

**Solution a** From Example 3 in Section 5.1 we have the repayment schedule using the amortization method as shown below.

Payment Number	Periodic Payment	Payment of Interest at $\frac{1}{2}\%$	Principal Repaid	Outstanding Balance
				\$6000.00
1	116.00	30.00	86.00	5914.00
2	116.00	29.57	86.43	5827.57
3	116.00	29.14	86.86	5740.71

**Solution b** Using the sum-of-digits approach we first calculate the total dollar value of all interest payments. We can find the smaller concluding payment  $X$  at the end of 5 years using the method of Section 5.1,

$$\begin{aligned} X &= 6000(1.005)^{60} - 116.00s_{\overline{59}|0.005}(1.005) \\ &= 8093.10 - 7977.32 = \$115.78 \end{aligned}$$

and then calculate the total interest in the loan as the difference between the total dollar value of all payments and the value of the loan; i.e.,

$$(59 \times 116.00 + 115.78) - 6000 = \$959.78$$

The sum of the digits from 1 to 60 is 1830. Thus,

$$\begin{aligned} \text{Interest in first payment} &= \frac{60}{1830} \times 959.78 = \$31.47 \\ \text{Interest in second payment} &= \frac{59}{1830} \times 959.78 = \$30.94 \\ \text{Interest in third payment} &= \frac{58}{1830} \times 959.78 = \$30.42 \end{aligned}$$

This leads to the following entries in our repayment schedule:

Payment Number	Periodic Payment	Amount of Interest	Principal Repaid	Outstanding Balance
				\$6000.00
1	116.00	31.47	84.53	5915.47
2	116.00	30.94	85.06	5830.41
3	116.00	30.42	85.58	5744.83

This illustrates the penalty involved in refinancing a loan, where the lending institution uses the sum-of-digits approximation. The penalty can be quite significant on long-term loans at high interest rates. In fact, it is possible under the sum-of-digits approximation for the interest portion of early payments to exceed the dollar size of these payments, which leads to an outstanding balance that is actually larger than the original loan (see Problem 7 in Exercise 5.4, Part A).

In the above illustration, if the consumer repaid the loan after one month the lending institution would earn \$31.47 of interest on their \$6000 over one month, or  $i = 0.5245\%$ . This corresponds to  $j_{12} = 6.294\%$  as opposed to  $j_{12} = 6\%$ .

We pointed out in Section 3.6 that most provinces in Canada have “Truth-in-Lending” Acts to protect the consumer. Unfortunately, these acts apply only if the loan is not renegotiated before the normal maturity date. Paying off a loan early constitutes “breaking the contract” and the Truth-in-Lending laws do not apply once the contract is broken.

If you have a consumer loan from a Canadian lending institution using the Rule of 78, to find the outstanding balance, the loans officer will not go through an entire repayment schedule but rather will use the method outlined in Example 2.

**EXAMPLE 2** To pay off the purchase of a mobile home, a couple takes out a loan of \$45 000 to be repaid with monthly payments over 10 years at  $j_{12} = 9\%$ . Using the sum-of-digits method find the outstanding debt at the end of 2 years.

**Solution** The monthly payment  $R$  required is

$$R = \frac{45\,000}{a_{\overline{120}|0.0075}} = \$570.05$$

The final payment  $X$  is

$$\begin{aligned} X &= 45\,000(1.0075)^{120} - 570.05s_{\overline{119}|0.0075}(1.0075) \\ &= 110\,311.07 - 109\,742.76 = \$568.31 \end{aligned}$$

$$\text{Total interest} = (119 \times 570.05 + 568.31) - 45\,000 = \$26\,404.26$$

The total interest to be repaid over the remaining 96 months is called the *interest rebate* and is calculated as

$$\frac{\text{sum of digits 1 to 96}}{\text{sum of digits 1 to 120}} \times 23\,404.26 = \frac{4656}{7260} \times 23\,404.26 = \$15\,009.67$$

The couple would be given the following figures:

Original total debt = $119R + X$	= 68 404.26
Less interest rebate	= 15 009.67
Less payments to date = $24 \times 570.05$	= 13 681.20
Outstanding balance at the end of 2 years	= \$39 713.39

It is interesting to note that the outstanding balance at the end of 2 years under the amortization method is

$$\begin{aligned} 45\,000(1.0075)^{24} - 570.05s_{\overline{24}|0.0075} &= 53\,838.61 - 14\,928.74 \\ &= \$38\,909.87 \end{aligned}$$

\* As of January 1, 1983, the Rule of 78 Method became illegal for consumer loans issued by federally chartered Canadian banks only.

**EXAMPLE 3** Prepare an Excel spreadsheet and show the first three months and last three months of a complete repayment schedule for the \$6000 loan of Example 1 using the sum-of-digits method. Also show total payments, total interest, and total principal paid.

**Solution:** We use the same headings in cells A1 through E1 as in the amortization schedule of Example 1 and we summarize the entries for line 0 and line 1 of an Excel spreadsheet.

CELL	ENTER
A2	0
E2	6,000
A3	=A2+1
B3	=116.00
C3	=((60-A3+1)/1830)*959.78
D3	=B3-C3
E3	=E2-D3

To generate the complete schedule copy A3.E3 to A4.E62

To get the last payment adjust B62 = E61+C62

To get the totals apply  $\Sigma$  to B1.D62

Below are the first 3 months and the last 3 months of a complete repayment schedule with required totals, using the sum-of-digits method.

	A	B	C	D	E
1	Pmt#	Payment	Interest	Principal	Balance
2	0				6,000.00
3	1	116.00	31.47	84.53	5,915.47
4	2	116.00	30.94	85.06	5,830.41
5	3	116.00	30.42	85.58	5,744.83

	A	B	C	D	E
1	Pmt#	Payment	Interest	Principal	Balance
2	58	116.00	1.57	114.43	230.31
3	59	116.00	1.05	114.95	115.26
4	60	115.78	0.52	115.26	-
5		6,843.78	959.78	6,000.00	

■

## Exercise 5.4

### Part A

1. A loan of \$900 is to be repaid with 6 equal monthly payments at  $j_{12} = 12\%$ . Find the monthly payment and construct the repayment schedule using the sum-of-digits method. (Compare this answer to Exercise 5.1, Part A, Question 4.)
2. A loan of \$1000 is to be repaid over one year with equal monthly payments at  $j_{12} = 9\%$ . Find the monthly payment and construct the repayment schedule using the sum-of-digits method.
3. To pay off the purchase of a car, Derek got a \$15 000 3-year loan at  $j_{12} = 9\%$ . He makes equal monthly payments. Find the outstanding balance on the loan just after the 24th payment using the sum-of-digits method.
4. Christine borrows \$20 000 to be repaid by monthly payments over 3 years at  $j_{12} = 10\frac{1}{2}\%$ . Find the outstanding balance at the end of 16 months using the sum-of-digits method.
5. Raymond is repaying a \$5000 loan at  $j_{12} = 15\%$  with monthly payments over 3 years. Just after the 12th payment he has the balance refinanced at  $j_{12} = 12\%$ . The balance is determined by the sum-of-digits method. If the number of payments remains unchanged, what will be the new monthly payments and what will be the monthly savings in interest? (Compare this to Exercise 5.3, Part A, Question 1.)
6. A 5-year \$6000 loan is to be repaid with monthly payments at  $j_{12} = 18\%$ . Just after making the 30th payment, the borrower has the balance refinanced at  $j_{12} = 12\%$  with the term of the loan to remain unchanged. If the balance is determined by the sum-of-digits method, what will be the monthly savings in interest? (Compare this to Exercise 5.3, Part A, Question 2.)
7. Consider a \$10 000 loan being repaid with monthly payments over 15 years at  $j_{12} = 15\%$ . Find the outstanding balance at the end of 2 years and at the end of 5 years using both the sum-of-digits method and the amortization method.
8. Michelle has a \$5000 loan that is being repaid by monthly payments over 4 years at  $j_{12} = 9\%$ . The lender uses the sum-of-digits method to determine outstanding balances. After 1 year of payments the lender's interest rate on new loans has dropped to  $j_{12} = 6\%$ . Will Michelle save money by refinancing the loan? (The term of the loan will not be changed.)

### Part B

1. Matthew can borrow \$15 000 at  $j_4 = 15\%$  and repay the loan with monthly payments over 10 years. If he wants to pay the loan off early, the outstanding balance will be determined using the sum-of-digits method.

He can also borrow \$15 000 with monthly payments over 10 years at  $j_4 = 16\%$  and pay the loan off at any time without penalty. The outstanding balance will be determined using the amortization method.

Matthew has an endowment insurance policy coming due in 4 years that could be used to pay off the outstanding balance at that time in full. Which loan should he take if he earns  $j_{12} = 6\%$  on his savings?

2. A loan of \$18 000 is to be repaid with monthly payments over 10 years at  $j_2 = 8\frac{1}{2}\%$ . Using the sum-of-digits method,
  - a) construct the first two and the last two lines of the repayment schedule;
  - b) find the interest and the principal portion of the 10th payment;

- c) find the outstanding balance at the end of 2 years and compare it with the outstanding balance at the same time calculated by the amortization method;
- d) advise whether the loan should be refinanced at the end of 2 years at current rate  $j_2 = 8\%$  with the term of the loan unchanged.
3. A \$20 000 home renovation loan is to be amortized over 10 years by monthly payments, with each regular payment rounded up to the next dollar, and the last payment reduced accordingly. Interest on the loan is at  $j_4 = 10\%$ . After 4 years the loan is fully paid off with an extra payment. Find the amount of this final payment if the sum-of-digits method is used to calculate the outstanding principal.

## Section 5.5

### Sinking Funds

When a specified amount of money is needed at a specified future date, it is a good practice to accumulate systematically a fund by means of equal periodic deposits. Such a fund is called a **sinking fund**. Sinking funds are used to pay off debts (see Section 5.6), to redeem bond issues, to replace worn-out equipment, to buy new equipment, or in one of the depreciation methods (see Section 7.4).

Since the amount needed in the sinking fund, the time the amount is needed and the interest rate that the fund earns are known, we have an annuity problem in which the size of the payment, the sinking-fund deposit, is to be determined. A schedule showing how a sinking fund accumulates to the desired amount is called a **sinking-fund schedule**.

**EXAMPLE 1** An eight-storey condominium apartment building consists of 146 two-bedroom apartment units of equal size. The Board of Directors of the Homeowners' Association estimated that the building will need new carpeting in the halls at a cost of \$25 800 in 5 years.

Assuming that the association can invest their money at  $j_{12} = 8\%$ , what should be the monthly sinking-fund assessment per unit?

**Solution** The sinking-fund deposits form an ordinary simple annuity with  $S = 25\,800$ ,  $i = \frac{2}{3}\%$ ,  $n = 60$ . We calculate the total monthly sinking-fund deposit

$$R = \frac{25\,800}{s_{\overline{60}|\frac{2}{3}\%}} = \$351.13 \text{ (rounded off)}$$

Per-unit assessment should be

$$\frac{351.13}{146} = \$2.41$$

■

**EXAMPLE 2** Show the first three lines and the last two lines of the sinking-fund schedule, explaining the growth of the fund in Example 1.

**Solution** At the end of the first month, a deposit of \$351.13 is made and the fund contains \$351.13. This amount earns interest at  $\frac{2}{3}\%$  for 1 month, i.e.,  $351.13 \times \frac{2}{300} = \$2.34$ . Thus the total increase at the end of the second month is

the second payment plus interest on the amount in the fund, i.e.,  $351.13 + 2.34 = \$353.47$ , and the fund will contain \$704.60. This procedure may be repeated to complete the entire schedule.

In order to complete the last two lines of the sinking-fund schedule without running the complete schedule, we may calculate the amount in the fund at the end of the 58th month as the accumulated value of 58 payments, i.e.,

$$351.13s_{\overline{58}|2/3\%} = \$24\,764.04$$

and complete the schedule from that point. The calculations are tabulated below.

End of the Month	Interest on Fund at $\frac{2}{3}\%$	Deposit	Increase in Fund	Amount in Fund
1	-	351.13	351.13	351.13
2	2.34	351.13	353.47	704.60
3	4.70	351.13	355.83	1060.43
⋮	⋮	⋮		
58				24 764.04
59	165.09	351.13	516.22	25 280.26
60	168.54	351.20*	519.67	25 800.00

\* The last deposit is adjusted to have the final amount in the fund equal \$25 800. ■

**EXAMPLE 3** Prepare an Excel spreadsheet and show the first 3 months and the last 3 months of the sinking-fund schedule of Example 2 above.

**Solution:** We reserve cells A1 through E1 for headings and summarize the entries in the table below.

	CELL	ENTER	INTERPRETATION
Headings	A1	Deposit #	'Deposit number' or 'End of the month #'
	B1	Interest	'Interest on fund'
	C1	Deposit	'Monthly deposit'
	D1	Increase	'Increase in fund'
	E1	Amount	'Amount in fund'
Line 0	A2	0	Time starts
	E2	0	Amount in fund at the beginning of month 1
Line 1	A3	=A2+1	End of month 1
	B3	=E2*(0.08/12)	Interest on fund at the end of month 1
	C3	351.13	Monthly deposit
	D3	=B3+C3	Increase in fund at the end of month 1
	E3	=E2+D3	Amount in fund at the end of month 1

To generate the complete schedule copy A3.E3 to A4.E62  
 To get the last deposit adjust C62 = E62-E61-B62

Below are the first 3 months and the last 3 months of a sinking-fund schedule

	A	B	C	D	E
1	Deposit#	Interest	Deposit	Increase	Amount
2	0				0
3	1	0	351.13	351.13	351.13
4	2	2.34	351.13	353.47	704.60
5	3	4.70	351.13	355.83	1,060.43

	A	B	C	D	E
1	Deposit#	Interest	Deposit	Increase	Amount
2	58	161.67	351.13	512.80	24,764.04
3	59	165.09	351.13	516.22	25,280.26
4	60	168.54	351.20	519.74	25,800.00

### Exercise 5.5

#### Part A

1. A couple is saving a down payment for a home. They want to have \$5000 at the end of 4 years in an account paying interest at  $j_1 = 6\%$ . How much must be deposited in the fund at the end of each year? Make out a schedule showing the growth of the fund.
2. A company wants to save \$100 000 over the next 5 years so they can expand their plant facility. How much must be deposited at the end of each year if their money earns interest at  $j_1 = 8\%$ ? Make out a schedule for this problem.
3. What quarterly deposit is required in a bank account to accumulate \$2000 at the end of 2 years if interest is at  $j_4 = 4\%$ ? Prepare a schedule for this problem.
4. A sinking fund earning interest at  $j_4 = 6\%$  now contains \$1000. What quarterly deposits for the next 5 years will cause the fund to grow to \$10 000? How much is in the fund at the end of 3 years?
5. A cottagers' association decides to set up a sinking fund to save money to have their cottage road widened and paved. They want to have \$250 000 at the end of 5 years and they can earn interest at  $j_1 = 9\%$ . What annual deposit is required per cottager if there are 30 cottages on the road? Show the complete schedule.
6. Find the quarterly deposits necessary to accumulate \$10 000 over 10 years in a sinking fund earning interest at  $j_4 = 6\%$ . Find the amount in the fund at the end of 9 years and complete the rest of the schedule.
7. A city needs to have \$200 000 at the end of 15 years to retire a bond issue. What annual deposits will be necessary if their money earns interest at  $j_1 = 7\%$ ? Make out the first three and last three lines of the schedule.
8. What monthly deposit is required to accumulate \$3000 at the end of 2 years in a bank account paying interest at  $j_4 = 10\%$ ?

9. A couple wants to save \$200 000 to buy some land. They can save \$3500 each quarter-year in a bank account paying  $j_4 = 9\%$ . How many years (to the nearest quarter) will it take them and what is the size of the final deposit?
10. In its manufacturing process, a company uses a machine that costs \$75 000 and is scrapped at the end of 15 years with a value of \$5000. The company sets up a sinking fund to finance the replacement of the machine, assuming no change in price, with level payments at the end of each year. Money can be invested at an annual effective interest rate of 4%. Find the value of the sinking fund at the end of the 10th year.

#### Part B

1. A homeowners' association decided to set up a sinking fund to accumulate \$50 000 by the end of 3 years to improve recreational facilities. What monthly deposits are required if the fund earns 5% compounded daily? Show the first three and the last two lines of the sinking-fund schedule.
2. Consider an amount that is to be accumulated with equal deposits  $R$  at the end of each interest period for 5 periods at rate  $i$  per period. Hence, the amount to be accumulated is  $Rs_{\overline{5}|i}$ . Do a complete schedule for this sinking fund. Verify that the sum-of-the-interest column plus the sum-of-the-deposit column equals the sum of the increase-in-the-fund column, and both sums equal the final amount in the fund.
3. A sinking fund is being accumulated at  $j_{12} = 6\%$  by deposits of \$200 per month. If the fund contains \$5394.69 just after the  $k$ th deposit, what did it contain just after the  $(k - 1)$ st deposit?

### Section 5.6

## The Sinking-Fund Method of Retiring a Debt

A common method of paying off long-term loans is to pay the interest on the loan at the end of each interest period and create a sinking fund to accumulate the principal at the end of the term of the loan. Usually, the deposits into the sinking fund are made at the same times as the interest payments on the debt are made to the lender. The sum of the interest payment and the sinking-fund payment is called the **periodic expense** or **cost of the debt**. It should be noted that the sinking fund remains under the control of the borrower. At the end of the term of the loan, the borrower returns the whole principal as a lump-sum payment by transferring the accumulated value of the sinking fund to the lender.

When the sinking-fund method is used, we define the **book value** of the borrower's debt at any time as the original principal minus the amount in the sinking fund. The book value of the debt may be considered as the outstanding balance of the loan.

**EXAMPLE** A city issues \$1 000 000 of bonds paying interest at  $j_2 = 9\frac{1}{8}\%$ , and by law it is required to create a sinking fund to redeem the bonds at the end of 8 years. If the fund is invested at  $j_2 = 8\%$ , find a) the semi-annual expense of the debt; b) the book value of the city's indebtedness at the beginning of the 7th year.

**Solution a**

Semi-annual interest payment on the debt:  $1\,000\,000 \times 0.045625 = \$45\,625$

Semi-annual deposit into the sinking fund:  $R = \frac{1\,000\,000}{s_{\overline{10}|0.04}} = \$45\,820$

Semi-annual expense of the debt  $= \$91\,445$

**Solution b** The amount in the sinking fund at the end of the 6th year is the accumulated value of the deposits; i.e.,

$$45\,820s_{\overline{12}|0.04} = \$688\,482.41$$

The book value of the city's indebtedness at the beginning of the 7th year is then

$$1\,000\,000 - 688\,482.41 = \$311\,517.59$$

## Exercise 5.6

**Part A**

1. A borrower of \$5000 agrees to pay interest semi-annually at  $j_2 = 10\%$  on the loan and to build up a sinking fund, which will repay the loan at the end of 5 years. If the sinking fund accumulates at  $j_2 = 4\%$ , find his total semi-annual expense. How much is in the sinking fund at the end of 4 years?
2. A city borrows \$250 000, paying interest annually on this sum at  $j_1 = 9\frac{1}{2}\%$ . What annual deposits must be made into a sinking fund earning interest at  $j_1 = 3\frac{1}{2}\%$  in order to pay off the entire principal at the end of 15 years? What is the total annual expense of the debt?
3. A company issues \$500 000 worth of bonds, paying interest at  $j_2 = 8\%$ . A sinking fund with semi-annual deposits accumulating at  $j_2 = 4\%$  is established to redeem the bonds at the end of 20 years. Find
  - a) the semi-annual expense of the debt;
  - b) the book value of the company's indebtedness at the end of the 15th year.
4. A city borrows \$2 000 000 to build a sewage treatment plant. The debt requires interest at  $j_2 = 10\%$ . At the same time, a sinking fund is established, which earns interest at  $j_2 = 4\frac{1}{2}\%$  to repay the debt in 25 years. Find
  - a) the semi-annual expense of the debt;
  - b) the book value of the city's indebtedness at the beginning of the 16th year.
5. On a debt of \$4000, interest is paid monthly at  $j_{12} = 12\%$  and monthly deposits are made into a sinking fund to retire the debt at the end of 5 years. If the sinking fund earns interest at  $j_4 = 3.6\%$ , what is the monthly expense of the debt?
6. On a debt of \$10 000, interest is paid semi-annually at  $j_2 = 10\%$  and semi-annual deposits are made into a sinking fund to retire the debt at the end of 5 years. If the sinking fund earns interest at  $j_{12} = 6\%$ , what is the semi-annual expense of the debt?
7. Interest at  $j_2 = 12\%$  on a loan of \$3000 must be paid semi-annually as it falls due. A sinking fund accumulating at  $j_4 = 8\%$  is established to enable the debtor to repay the loan at the end of 4 years.
  - a) Find the semi-annual sinking fund deposit and construct the last two lines of the sinking fund schedule, based on semi-annual deposits.
  - b) Find the semi-annual expense of the loan.
  - c) What is the outstanding principal (book value of the loan) at the end of 2 years?

8. A 10-year loan of \$10 000 at  $j_1 = 11\%$  is to be repaid by the sinking-fund method, with interest and sinking fund payments made at the end of each year. The rate of interest earned in the sinking fund is  $j_1 = 5\%$ . Immediately after the 5th year's payment, the lender requests that the outstanding principal be repaid in one lump sum. Calculate the amount of extra cash the borrower has to raise in order to extinguish the debt.

**Part B**

1. A company issues \$2 000 000 worth of bonds paying interest at  $j_{12} = 10\frac{1}{2}\%$ . A sinking fund accumulating at  $j_4 = 6\%$  is established to redeem the bonds at the end of 15 years. Find
  - a) the monthly expense of the debt;
  - b) the book value of the company's indebtedness at the beginning of the 6th year.
2. A man is repaying a \$10 000 loan by the sinking-fund method. His total monthly expense is \$300. Out of this \$300, interest is paid to the lender at  $j_{12} = 12\%$  and a deposit is made to a sinking fund earning  $j_{12} = 9\%$ . Find the duration of the loan and the final smaller payment.
3. A \$100 000 loan is to be repaid in 15 years, with a sinking fund accumulated to repay principal *plus interest*. The loan charges  $j_2 = 12\%$ , while the sinking fund earns  $j_2 = 5\%$ . What semi-annual sinking fund deposit is required?
4. A loan of \$20 000 bears interest on the amount outstanding at  $j_1 = 10\%$ . A deposit is to be made in a sinking fund earning interest at  $j_1 = 4\%$ , which will accumulate enough to pay one-half of the principal at the end of 10 years. In addition, the debtor will make level payments to the creditor, which will pay interest at  $j_1 = 10\%$  on the outstanding balance first and the remainder will repay the principal. What is the total annual payment, including that made to the creditor and that deposited in the sinking fund, if the loan is to be completely retired at the end of 10 years?
5. John borrows \$10 000 for 10 years and uses a sinking fund to repay the principal. The sinking-fund deposits earn an annual effective interest rate of 5%. The total required payment for both the interest and the sinking-fund deposit made at the end of each year is \$1445.05. Calculate the annual effective interest rate charged on the loan.
6. A company borrows \$10 000 for five years. Interest of \$600 is paid semi-annually. To repay the principal of the loan at the end of 5 years, equal semi-annual deposits are made into a sinking fund that credits interest at a nominal rate of 8% compounded quarterly. The first payment is due in 6 months. Calculate the annual effective rate of interest that the company is paying to service and retire the debt.
7. On August 1, 1993, Mrs. Chan borrows \$20 000 for 10 years. Interest at 11% per annum convertible semi-annually must be paid as it falls due. The principal is replaced by means of level deposits on February 1 and August 1 in years 1994 to 2003 (inclusive) into a sinking fund earning  $j_1 = 7\%$  in 1994 through December 31, 1998, and  $j_1 = 6\%$  January 1, 1999, through 2003.
  - a) Find the semi-annual expense of the loan.
  - b) How much is in the sinking fund just after the August 1, 2002, deposit?
  - c) Show the sinking-fund schedule entries at February 1, 2003, and August 1, 2003.
8. Mr. White borrows \$15 000 for 10 years. He makes total payments, annually, of \$2000. The lender receives  $j_1 = 10\%$  on his investment each year for the first 5 years and  $j_1 = 8\%$  for the second 5 years. The balance of each payment is invested in a sinking fund earning  $j_1 = 7\%$ .

- a) Find the amount by which the sinking fund is short of repaying the loan at the end of 10 years.
- b) By how much would the sinking-fund deposit (in each of the first 5 years only) need to be increased so that the sinking fund at the end of 10 years will be just sufficient to repay the loan?

### Section 5.7

## Comparison of Amortization and Sinking-Fund Methods

We have discussed the two most common methods of paying off long-term loans: the amortization method and the sinking-fund method. When there are several sources available from which to borrow money, it is important to know how to compare the available loans and choose the cheapest one. The borrower should choose that source for which the periodic expense of the debt is the lowest. When the amortization method is used, the periodic expense of the debt is equal to the periodic amortization payment. When the sinking-fund method is used, the periodic expense of the debt is the sum of the interest payment and the sinking-fund deposit.

To study the relationship between the amortization and sinking-fund methods, we define the following:

$L$  = principal of the loan

$n$  = number of interest periods during the term of the loan

$i_1$  = loan rate per interest period using amortization

$i_2$  = loan rate per interest period using the sinking fund

$i_3$  = sinking-fund rate per interest period

$E_A$  = periodic expense using amortization =  $\frac{L}{a_{\overline{n}|i_1}} = L \left( \frac{1}{s_{\overline{n}|i_1}} + i_1 \right)$   
(See Exercise 3.3B 1 b.)

$E_S$  = periodic expense using sinking fund =  $\frac{L}{s_{\overline{n}|i_3}} + Li_2 = L \left( \frac{1}{s_{\overline{n}|i_3}} + i_2 \right)$

We shall examine the relationship between  $E_A$  and  $E_S$  for different levels of rates  $i_1, i_2, i_3$ .

a) Let  $i_1 = i_2 = i_3 \equiv i$ , then  $E_A = E_S$

b) Let  $i_3 < i_1 = i_2$ , then  $i_3 < i_1$  implies

$$\begin{aligned} s_{\overline{n}|i_3} &< s_{\overline{n}|i_1} \\ \frac{1}{s_{\overline{n}|i_1}} &< \frac{1}{s_{\overline{n}|i_3}} \\ \frac{1}{s_{\overline{n}|i_1}} + i_1 &< \frac{1}{s_{\overline{n}|i_3}} + i_2 \\ E_A &< E_S \end{aligned}$$

c) Let  $i_3 > i_1 = i_2$ , then  $i_3 > i_1$  implies

$$\begin{aligned} s_{\overline{n}|i_3} &> s_{\overline{n}|i_1} \\ \frac{1}{s_{\overline{n}|i_1}} &> \frac{1}{s_{\overline{n}|i_3}} \\ \frac{1}{s_{\overline{n}|i_1}} + i_1 &> \frac{1}{s_{\overline{n}|i_3}} + i_2 \\ E_A &> E_S \end{aligned}$$

d) Let  $i_3 < i_1 < i_2$ , then  $i_3 < i_1$  implies

$$\frac{s_{\overline{m}|i_3}}{s_{\overline{m}|i_1}} < \frac{s_{\overline{m}|i_1}}{s_{\overline{m}|i_3}}$$

$$\frac{1}{s_{\overline{m}|i_1}} + i_1 < \frac{1}{s_{\overline{m}|i_3}} + i_2$$

$$E_A < E_S$$

e) Let  $i_3 < i_2 < i_1$ . In this case, which is the most common, we can't tell which method is cheaper. We must calculate the actual periodic costs to determine the cheaper source of money, i.e., the source with the least periodic expense.

**EXAMPLE 1** A company wishes to borrow \$500 000 for 5 years. One source will lend the money at  $j_2 = 12\%$  if it is amortized by semi-annual payments. A second source will lend the money at  $j_2 = 11\%$  if only the interest is paid semi-annually and the principal is returned in a lump sum at the end of 5 years. If the second source is used, a sinking fund will be established by semi-annual deposits that accumulate at  $j_{12} = 6\%$ . How much can the company save semi-annually by using the better plan?

**Solution** When the first source is used, the semi-annual expense of the debt is

$$E_A = \frac{500\,000}{a_{\overline{10}|0.06}} = \$67\,933.98$$

When the second source is used, the interest on the debt paid semi-annually is  $5\frac{1}{2}\%$  of 500 000 = \$27 500.00.

To calculate the semi-annual deposit into the sinking fund, we must first calculate the semi-annual rate  $i$  equivalent to  $j_{12} = 6\%$

$$(1+i)^2 = (1.005)^{12}$$

$$i = (1.005)^6 - 1$$

$$i = .030377509$$

Now we calculate the semi-annual deposit  $R$  into the sinking fund

$$R = \frac{500\,000}{s_{\overline{10}|i}} = \$43\,539.48$$

The semi-annual expense of the debt using the second source is

$$E_S = 27\,500 + 43\,539.48 = \$71\,039.48$$

Thus, the first source, using amortization, is cheaper and the company can save  $71\,039.48 - 67\,933.98 = \$3105.50$  semi-annually. ■

**EXAMPLE 2** A firm wants to borrow \$500 000. One source will lend the money at  $j_4 = 10\%$  if interest is paid quarterly and the principal is returned in a lump sum at the end of 10 years. The firm can set up a sinking fund at  $j_4 = 7\%$ . At what rate  $j_4$  would it be less expensive to amortize the debt over 10 years?

**Solution** We calculate the quarterly expense of the debt.

$$\begin{aligned} \text{Interest payment:} & \quad 500\,000 \times .025 = \$12\,500.00 \\ \text{Sinking-fund deposit:} & \quad \frac{500\,000}{s_{\overline{40}|.0175}} = \$8\,736.05 \\ \text{Quarterly expense:} & \quad \underline{\hspace{1.5cm}} = \$21\,236.05 \end{aligned}$$

The amortization method will be as expensive if the quarterly amortization payment is equal to \$21 236.05. Thus, we want to find the interest rate  $i$  per quarter (and then  $j_4$ ) given  $A = 500\,000$ ,  $R = 21\,236.05$ ,  $n = 40$ .

We have

$$\begin{aligned} 500\,000 &= 21\,236.05 a_{\overline{40}|i} \\ a_{\overline{40}|i} &= 23.544868 \end{aligned}$$

We want to find the rate  $j_4 = 4i$  such that  $a_{\overline{40}|i} = 23.5449$ . A starting value to solve  $a_{\overline{40}|i} = 23.5449$  is

$$i = \frac{1 - \left(\frac{23.5449}{40}\right)^2}{23.5449}$$

or  $j_4 = 4i = 11.10\%$ . Using linear interpolation, we calculate

$a_{\overline{40} i}$	$j_4$	
.9633	24.0781	11%
.5332	23.5449	$j_4$
	23.1148	12%

$\left. \begin{array}{l} \left. \left. \begin{array}{l} 24.0781 \\ 23.5449 \\ 23.1148 \end{array} \right\} \right\} d \\ \left. \left. \begin{array}{l} 11\% \\ j_4 \\ 12\% \end{array} \right\} \right\} 1\% \end{array} \right\} \begin{array}{l} \frac{d}{1\%} = .5332 \\ d \doteq .55\% \\ \text{and } j_4 = 11.55\% \end{array}$

If the firm can borrow the money and amortize the debt at less than  $j_4 = 11.55\%$ , then it will be less expensive than a straight loan at  $j_4 = 10\%$  with a sinking fund at  $j_4 = 7\%$ . ■

### Exercise 5.7

**Part A**

1. A company borrows \$50 000 to be repaid in equal annual instalments at the end of each year for 10 years. Find the total annual cost under the following conditions:
  - a) the debt is amortized at  $j_1 = 9\%$ ;
  - b) interest at 9% is paid on the debt and a sinking fund is set up at  $j_1 = 9\%$ ;
  - c) interest at 9% is paid on the debt and a sinking fund is set up at  $j_1 = 6\%$ .
2. A company can borrow \$180 000 for 15 years. They can amortize the debt at  $j_1 = 10\%$ , or they can pay interest on the loan at  $j_1 = 9\%$  and set up a sinking fund at  $j_1 = 7\%$  to repay the loan. Which plan is cheaper and by how much per annum?
3. A firm wants to borrow \$60 000 to be repaid over 5 years. One source will lend them the money at  $j_2 = 10\%$  if it is amortized by semi-annual payments. A second source will lend them money at  $j_2 = 9\frac{1}{2}\%$  if only the interest is paid semi-annually and the principal is returned in a lump sum at the end of 5 years. The firm can earn  $j_2 = 4\%$  on their savings. Which source should be used for the loan and how much will be saved each half-year?

4. A company can borrow \$100 000 for 10 years by paying the interest as it falls due at  $j_2 = 9\%$  and setting up a sinking fund at  $j_2 = 7\%$  to repay the debt. At what rate  $j_2$  would an amortization plan have the same semi-annual cost?
5. A city can borrow \$500 000 for 20 years by issuing bonds on which interest will be paid semi-annually at  $j_2 = 9\frac{1}{8}\%$ . The principal will be paid off by a sinking fund consisting of semi-annual deposits invested at  $j_2 = 8\%$ . Find the nominal rate  $j_2$  at which the loan could be amortized at the same semi-annual cost.
6. A firm can borrow \$200 000 at  $j_1 = 9\%$  and amortize the debt for 10 years. From a second source, the money can be borrowed at  $j_1 = 8\frac{1}{2}\%$  if the interest is paid annually and the principal is repaid in a lump sum at the end of 10 years. What yearly rate  $j_1$  must the sinking fund earn for the annual expense to be the same under the two options?
7. A company wants to borrow \$500 000. One source of funds will agree to lend the money at  $j_4 = 8\%$  if interest is paid quarterly and the principal is paid in a lump sum at the end of 15 years. The firm can set up a sinking fund at  $j_4 = 6\%$  and will make quarterly deposits.
  - a) What is the total quarterly cost of the loan?
  - b) At what rate  $j_4$  would it be less expensive to amortize the debt over 15 years?
8. You are able to repay an \$80 000 loan by either a) amortization at  $j_{12} = 7\%$  with 12 months payments; or b) at  $j_{12} = 6\frac{1}{2}\%$  using a sinking fund earning  $j_{12} = 4\%$ , and paid off in 1 year. Which method is cheaper?

#### Part B

1. A company needs to borrow \$200 000 for 6 years. One source will lend them the money at  $j_2 = 10\%$  if it is amortized by monthly payments. A second source will lend the money at  $j_4 = 9\%$  if only the interest is paid monthly and the principal is returned in a lump sum at the end of 6 years. The company can earn interest at  $j_{365} = 6\%$  on the sinking fund. Which source should be used for the loan and how much will be saved monthly?
2. Tanya can borrow \$10 000 by paying the interest on the loan as it falls due at  $j_2 = 12\%$  and by setting up a sinking fund with semi-annual deposits that accumulate at  $j_{12} = 9\%$  over 10 years to repay the debt. At what rate  $j_4$  would an amortization scheme have the same semi-annual cost?
3. A loan of \$100 000 at 8% per annum is to be repaid over 10 years; \$20 000 by the amortization method and \$80 000 by the sinking-fund method, where the sinking fund can be accumulated with annual deposits at  $j_4 = 5\%$ . What extra annual payment does the above arrangement require as compared to repayment of the whole loan by the amortization method?
4. A company wants to borrow a large amount of money for 15 years. One source would lend the money at  $j_2 = 9\%$ , provided it is amortized over 15 years by monthly payments. The company could also raise the money by issuing bonds paying interest semi-annually at  $j_2 = 8\frac{1}{2}\%$  and redeemable at par in 15 years. In this case, the company would set up a sinking fund to accumulate the money needed for the redemption of the bonds at the end of 15 years. What rate  $j_{12}$  on the sinking fund would make the monthly expense the same under the two options?
5. A \$10 000 loan is being repaid by the sinking-fund method. Total annual outlay (each year) is \$1400 for as long as necessary, plus a smaller final payment made 1 year after the last regular payment. If the lender receives  $j_1 = 8\%$  and the sinking fund accumulates at  $j_1 = 6\%$ , find the time and amount of the last irregular final payment.

6. A \$5000 loan can be repaid quarterly for 5 years using amortization and an interest rate of  $j_{12} = 10\%$  or by a sinking fund to repay both principal and accumulated interest. If paid by a sinking fund, the interest on the loan will be  $j_{12} = 9\%$ . What annual effective rate must the sinking fund earn to make the quarterly cost the same for both methods?

## Section 5.8

## Summary and Review Exercises

- Outstanding balance  $B_k$  (immediately after the  $k$ th payment has been made) by the retrospective method (looking back):

$$B_k = A(1+i)^k - Rs\bar{s}_{\overline{k}|i}$$

by the prospective method (looking ahead) assuming all payments equal

$$B_k = Ra_{\overline{n-k}|i}$$

- Total interest = Total payments – Amount of Loan
- For a loan of  $\$A$  to be amortized with level payments of  $\$R$  at the end of each period for  $n$  periods, at rate  $i$  per period, in the  $k$ th line of the amortization schedule ( $1 \leq k \leq n$ ):  
Interest payment  $I_k = R[1 - (1+i)^{-(n-k+1)}]$   
Principal payment  $P_k = R(1+i)^{-(n-k+1)}$   
Outstanding balance  $B_k = Ra_{\overline{n-k}|i}$

Successive principal payments are in the ratio  $1+i$ , that is  $\frac{P_{k+1}}{P_k} = (1+i)$ .

- In the sum-of-digits approximation to the amortization method, the interest portion of the  $k$ th payment is given by

$$I_k = \frac{n-k+1}{s} I$$

where  $n$  = number of payments

$$s = \text{sum of digits from 1 to } n = \frac{n(n+1)}{2}$$

$I$  = total interest

- For a sinking fund designed to accumulate a specified amount of  $\$S$  by equal deposits of  $\$R$  at the end of each period for  $n$  periods, at rate  $i$  per period, in the  $k$ -th line of the sinking-fund schedule ( $1 \leq k \leq n$ ):

$$\text{Interest on fund} = iRs_{\overline{k-1}|i} = R[(1+i)^{k-1} - 1]$$

$$\text{Increase in fund} = R + R[(1+i)^{k-1} - 1] = R(1+i)^{k-1}$$

$$\text{Amount in fund} = Rs\bar{s}_{\overline{k}|i}$$

- When a loan is paid off by the sinking-fund method, the borrower pays interest on the loan at the end of each period and accumulates the principal of the loan in a sinking fund. The principal of the loan is repaid at the end of the term of the loan as a lump sum from the sinking fund.

Periodic expense of the loan = Interest payment + Sinking-fund deposit.

Book value of the loan after  $k$  periods = Principal of the loan – Amount in the sinking fund.

## Review Exercises 5.8

1. The Roberts borrow \$15 000 to be repaid with monthly payments over 10 years at  $j_{12} = 15\%$ .
  - a) Find the monthly payment required.
  - b) Find the outstanding balance of the loan after three years (36 payments) and split the 37th payment into principal and interest under
    - i) the amortization method;
    - ii) the sum-of-digits method.
2. A loan of \$20 000 is to be amortized by 20 quarterly payments over 5 years at  $j_{12} = 7\frac{1}{2}\%$ . Split the 9th payment into principal and interest.
3. A loan of \$10 000 is repaid by 5 equal annual payments at  $j_2 = 14\%$ . What is the total amount of interest paid?
4. A company wants to borrow a large sum of money to be repaid over 10 years. The company can issue bonds paying interest at  $j_2 = 8\frac{1}{2}\%$  redeemable at par in 10 years. A sinking fund earning  $j_{12} = 7\frac{1}{2}\%$  can be used to accumulate the amount needed in 10 years to redeem the bonds. At what rate  $j_2$  would the semi-annual cost be the same if the debt were amortized over 10 years?
5. Interest at  $j_2 = 12\%$  on a debt of \$3000 must be paid as it falls due. A sinking fund accumulating at  $j_4 = 8\%$  is established to enable the debtor to repay the loan at the end of four years. Find the semi-annual sinking-fund deposit and construct the last two lines of the sinking-fund schedule.
6. For a \$60 000 mortgage at  $j_2 = 10\%$  amortized over 25 years, find
  - a) the level monthly payment required;
  - b) the outstanding balance just after the 48th payment;
  - c) the principal portion of the 49th payment; and
  - d) the total interest paid in the first 48 payments.
7. A \$10 000 loan is being amortized over 12 years with monthly payments at  $j_2 = 8\%$ . After 5 years you want to renegotiate the loan. You discover that the outstanding balance will be calculated using the sum-of-digits method. You can refinance by borrowing the money needed to pay off the outstanding balance at  $j_2 = 7\%$  and repay this new loan by accumulating the total principal and interest that will be due in 7 years in a sinking fund earning  $j_{12} = 6\%$ . Should you refinance?
8. As part of the purchase of a home on January 1, 1996, you negotiated a mortgage in the amount of \$110 000. The amortization period for calculation of the level payments (principal and interest) was 25 years and the initial interest rate was 6% per annum compounded semi-annually.
  - a) What was the initial monthly payment?
  - b) During 1996-2000 inclusive (and January 1, 2001) all monthly mortgage payments were made as they became due. What was the balance of the loan owing just after the payment made January 1, 2001?
  - c) At January 1, 2001 (just after the payment then due) the loan was renegotiated at 8% per annum compounded semi-annually (with the end date of the amortization period unchanged). What was the new monthly payment?
  - d) All payments, as above, have been faithfully made. How much of the September 1, 2001, payment will be principal and how much represents interest?
9. A couple has a \$150 000, 5-year mortgage at  $j_2 = 9\%$  with a 20-year amortization period. After exactly 3 years (36 payments) they could renegotiate a new mortgage at  $j_2 = 7\%$ . If the bank charges an interest penalty of 3 times the monthly interest due on the outstanding balance at the time of renegotiation, what will their new monthly payment be?

10. Janet wants to borrow \$10 000 to be repaid over 10 years. From one source, money can be borrowed at  $j_1 = 10\%$  and amortized by annual payments. From a second source, money can be borrowed at  $j_1 = 9\frac{1}{2}\%$  if only the interest is paid annually and the principal repaid at the end of 10 years. If the second source is used, a sinking fund will be established by annual deposits that accumulate at  $j_4 = 8\%$ . How much can Janet save annually by using the better plan?
11. Given the following information, find the original value of the loan.

Payment	Interest Paid	Principal Repaid
1		10.00
2	389.00	
3		
4		13.31

12. A loan is paid off over 19 years with equal monthly payments. The total interest paid over the life of the loan is \$5681.17. Using the sum-of-digits method, determine the amount of interest paid in the 163rd payment.
13. The XYZ Mortgage Company lends you \$100 000 at 9% per annum convertible semi-annually. The loan is to be repaid by monthly payments at the end of each month for 20 years and the rate is guaranteed for 5 years.
- Find the monthly payment.
  - Find the total amount of interest paid over the first 5 years.
  - Split the first monthly payments into principal and interest portions
    - based on the true amortization method, and
    - based on the sum-of-digits approximation method.
  - If after 5 years of payments interest rates have increased to 11% per annum convertible semi-annually, find the new monthly payment at time of mortgage renegotiation exactly 5 years after the original loan agreement based on the true amortization method.
14. A loan of \$2000 is being repaid by equal monthly payments for an unspecified length of time. Interest on the loan is  $j_{12} = 15\%$ .
- If the amount of principal in the 4th payment is \$40, what amount of the 18th payment will be principal?
  - Find the regular monthly payment.
15. On a loan at  $j_{12} = 12\%$  with monthly payments, the amount of principal in the 8th payment is \$62.
- Find the amount of principal in the 14th payment.
  - If there are 48 equal payments in all, find the amount of the loan.
16. Given the following part of an amortization schedule, find X.

Year	Payment	Interest	Principal	Balance
1		50 000	180 975	819 025
2				
3				X

17. A couple purchases a home worth \$150 000 by paying \$30 000 down and taking out a 5-year mortgage for \$120 000 at  $j_2 = 10.25\%$ . The mortgage will be amortized over 25 years with equal monthly payments. How much of the principal is repaid during the first year?

18. You take out an \$80 000 mortgage at  $j_2 = 9\%$  with a 25-year amortization period.
- Find the monthly payment required, rounded up to the dime.
  - Find the reduced final payment.
  - Find the total interest paid during the 4th year.
  - At the end of 4 years, you pay down an additional \$2500 (no penalty).
    - How much sooner will the mortgage be paid off?
    - What would be the difference in total payments over the life of the mortgage?
19. You take out a car loan of \$10 000 at  $j_{12} = 12\%$ . It is to be paid off by monthly payments for 50 months. Payments are rounded up to the next dollar and the final payment is reduced accordingly. After 15 months you decide to pay off the loan. How much is outstanding after your 15th payment if the sum-of-digits method is used?
20. A company decides to borrow \$100 000 at  $j_1 = 12\%$  in order to finance a new equipment purchase. One of the conditions of the loan is that the company must make annual payments into a sinking fund (the sinking fund will be used to pay off the loan at the end of 20 years). The sinking-fund investment will earn  $j_1 = 6\%$ .
- What is the amount of each sinking-fund payment if they are all to be equal?
  - What is the total annual cost of the loan?
  - What overall annual effective compound interest rate is the company paying to borrow the \$100 000 when account is taken of the sinking-fund requirement?
21. A \$50 000 loan at  $j_1 = 7\%$  is to be amortized over 15 years by annual payments.
- Find the regular payment and the reduced final payment.
  - The borrower accumulates the money for each annual payment by making 12 monthly deposits into a sinking fund earning  $j_{12} = 6\%$ . Find the size of each deposit for the first 14 years.
  - If the sum-of-digits method is used to calculate the outstanding principal, what is the outstanding balance after 6 years?
22. A loan of \$15 000 is to be paid off over 12 years by equal monthly payments at  $j_{12} = 18\%$ .
- Compare the outstanding balance at the end of 2 years using the amortization and the sum-of-digits methods.
  - Should the borrower refinance the loan without penalty at the end of 2 years at  $j_{12} = 16\%$  if the lender uses the sum-of-digits method?

## CASE STUDY I

### *Comparison of amortization, sum-of-digits and sinking-fund methods*

A \$10 000 loan at  $j_{12} = 12\%$  is to be paid off by monthly payments for 5 years.

Using

- the amortization method,
- the sum-of-digits method,
- the sinking-fund method, with  $j_{365} = 6\frac{1}{2}\%$  on the sinking fund, calculate and compare
  - the monthly expense of the loan;
  - the outstanding balance of the loan at the end of 2 years; and
  - the interest and principal payment at the end of the 1st month and at the end of 2 years.

**CASE STUDY II****Increasing extra annual payments**

A \$100 000 mortgage is taken out at  $j_2 = 8\%$ , to be amortized over 25 years by monthly payments. Payments are rounded up to the *next dime* and the final payment is reduced accordingly.

- Find the regular monthly payment and the reduced final payment.
- Suppose extra payments are made at the end of every year to get the mortgage paid off sooner. Find the time and amount of the last payment on the mortgage if these extra payments are:  
\$300 at the end of year 1, \$350 at the end of year 2, \$400 at the end of year 3, \$450 at the end of year 4, ... (increasing by \$50 each year)

**CASE STUDY III****Mortgage amortization**

A \$90 000 mortgage at  $j_2 = 9\%$  is amortized over 25 years by monthly payments.

- Find the regular monthly payment and the smaller final payment.
- Find the total interest (total cost of financing).
- Show the first three lines of the amortization schedule.
- Find the outstanding balance after 2 years.
- Find the total principal and the total interest paid in the first 2 years.
- Suppose you paid an extra \$1000 after 2 years. How much interest would this save over the life of the mortgage?
- How much less interest would be paid if the mortgage could be amortized over 20 years rather than 25 years?
- In the 25-year mortgage, after 2 years, interest rates drop to  $j_2 = 8.5\%$ . There is a penalty of 3 month's interest on the outstanding balance for early repayment. Does it pay to refinance?

**CASE STUDY IV****Accelerated mortgage payments**

“Invest” that extra money back into your mortgage. (from Royal Trust)

Does it pay to accelerate mortgage payments in a low interest rate environment?

Interest rates have dropped so low, relative to what they were a few years ago, that you have to wonder if there's any merit to stepping up the mortgage payments on your home. After all, there's a temptation to invest—or spend—the difference between today's payments and the ones you made a few years ago. With the stock market so hot, you may be tempted to try to make a buck or two from stocks or mutual funds.

Forget it. Your humble, terribly dull mortgage is a far better investment.

The reason: our tax system will force you to pay tax on any earnings from your investment. If you pay down your mortgage, there's no tax on the interest you save.

“Whether the interest rate of your mortgage is 6 per cent or 12 per cent, the best investment is to pay off your mortgage,” says Tom Alton, president of Bank of Montreal Mortgage Corp.

Jack Quinn, president of CIBC Mortgage Corp., agrees. “On an after-tax basis, it's pretty hard to find something that's as good as a mortgage.” Someone in a 50 per cent tax bracket would have to earn at least 16 per cent just to net 8 per cent after tax on an investment.

Most lenders offer a number of strategies to save you money.

- Provided you can afford it, consider a shorter amortization period when your mortgage comes up for renewal. For instance, if you had a 11 per cent mortgage and renewed at 8 per cent, continue to pay at the old rate, instead of the reduced new one. It will shorten the amortization because the difference will be applied to the principal.

Assuming you had a 25-year \$100,000 mortgage at 11 per cent, your monthly payment would be \$962.50. If the rate drops to 8 per cent, the payment would be \$763.20, for a difference of \$199.30 a month.

What difference is another couple of hundred bucks going to make? Plenty. If you continue to make the old monthly payments, you can reduce the amortization to 14.6 years, according to Mr. Alton.

- Another strategy is to take advantage of any opportunity to make larger monthly payments. For instance, under Bank of Montreal's "10 plus 10" plan, you can increase your monthly payments by up to 10 percent. Royal Trust goes a bit further and allows borrowers to "double up" their monthly mortgage payments—that is, pay up to an additional 100 per cent of the payment any or every month of the year (see accompanying chart).
- Most institutions let you make an annual lump-sum payment of up to 10 per cent of the original principal. Canada Trust has a higher maximum: 15 per cent.
- Finally, consider an accelerated weekly mortgage. Take your monthly payment, divide it by four, and pay that amount on a weekly basis. That means that you will be painlessly paying the equivalent of 13 months in the space of a year. "If you do that you'll make larger payments, but knock about seven years of payments off a 25-year mortgage," Mr. Alton says.

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### HOW TO SAVE ON MORTGAGES

\$100,000 mortgage, 8% interest rate

#### By shortening the term:

Amortization	Payment	Interest paid over life of mortgage	Interest saved versus 25-year amortization
25 years	\$763	\$129,098.54	--
20 years	\$828	\$98,927.31	\$30,171.23
15 years	\$948	\$70,692.37	\$58,406.17
10 years	\$1,206	\$44,794.25	\$84,304.29

#### By doubling-up payments

	Interest paid over life of mortgage	Interest saved versus no double-up payments	Time paid off sooner
No double-up payments	\$129,098.54	--	
1 double-up payment a year (month 4)	\$98,863.53	\$30,285.06	4 years and 11 months
2 double-up payments a year (months 4, 10)	\$81,925.83	\$47,172.71	7 years and 11 months
4 double-up payments a year (months 3, 6, 9, 12)	\$61,486.75	\$67,611.69	11 years and 9 months

Data: Royal Trust

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- Verify the figures in the first 2 lines of the first table (i.e., by shortening the term).
- Verify the figures in the second and fourth line of the second table (i.e., by doubling up payments).