Section 3:
Trend Analysis

Outline and Source

- **General Concept**
- **Parametric and Non-Parametric Trend Tests**
- **Accounting for Serial Correlation**
- **Trend Analysis for Count Data**
- **Additional Remarks**

- **Material primarily based on:**
  - Yue et al. 2002, Zhang & Zwiers 2004
  - Frei & Schär 2001
  - …

General Concept

- **Data series = trend + stochastic variations**

  $x(t), \quad t = t_1, .., t_n,$ the data series

  $x(t) = f(t) + \varepsilon(t)$ $\quad f(t):$ a deterministic function in time

  $\varepsilon(t):$ stochastic variations, a random variable

- **Null hypothesis and alternative:**

  H_0: $f(t)$ is constant (i.e. not dependent on time)

  H_A: $f(t)$ is not constant

- **Test:**

  Construct a suitable test statistic T that can distinguish between H_0 and H_A, in the presence of $\varepsilon(t)$

General Concept

- **Assumptions**
  - Every trend analysis makes assumptions about $\varepsilon(t)$.
  - Whether or not the data satisfies these, needs verification.
  - If assumptions are violated, test results can be misleading.

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Trend Analysis

Parametric & Non-Parametric Tests

Linear Regression

parametric test

- **Concept: Linear trend + normal white noise**

$$x(t) = \beta \cdot t + \alpha + \varepsilon(t) \qquad \varepsilon(t) \approx N(0,\sigma), \quad \sigma(t) = \sigma$$

➨ Ordinary least squares regression

$$\hat{\beta} = r_{xt}\frac{s_x}{s_t} \quad \hat{\alpha} = \bar{x} - \hat{\beta}\cdot\bar{t} \qquad s_x^2 = \frac{1}{n-1}\sum_i (x_i - \bar{x})^2 \quad \text{dito } s_t$$

- **Null hypothesis and alternative:**

H_0: $\beta = 0$ (i.e. data does not depend linearly on time)

H_A: $\beta \neq 0$

- **Test statistic:**

$$T = \frac{\hat{\beta}}{\hat{\sigma}_\beta} \qquad \hat{\sigma}_\beta^2 = \frac{\sigma_{res}^2}{\sum_i (t_i - \bar{t})^2}$$

$\hat{\sigma}_\beta$ standard error of slope

σ_{res} residual variance

Linear Regression

- **Null distribution**
  - T is distributed like students's T with n-2 degrees of freedom, if:
    (a) residuals are independent (i.e. no serial correlation) and
    (b) residuals are normally distributed
    (c) residual variance is constant
- **Verify assumptions**
  - Normality of residuals in histogram and Q-Q plot
  - Equal variance by Tukey-Anscombe plot (residuals vs. fitted)
  - Independence by auto-correlation function
  - Data transformations may be used to ensure normality of residuals and constant variance
  - T-Test is sensitive to outliers and not robust

`R: lm, coef, resid, summary`

Linear Regression

- **Example**

$\beta = 0.1 \cdot m^3 \cdot s^{-1} / a$

$T = 2.38$

$q_T(0.025) = -1.99$
$q_T(0.975) = +1.99$
68 degrees of freedom

p-value: 0.02

➨ Trend is statistically significant, i.e. evidence for change

Lineare Regression

Residual Analysis:

Maximum winter-time runoff, Birs-Moutier

1931-2000

Theil-Sen Trend Estimate

- **A robust trend estimator for a linear trend**

$x_i \quad i = 1,..,n$ regular time series

$$\beta_{TS} = \underset{i<k}{\text{median}}\left(\frac{x_k - x_i}{k - i}\right)$$

- Median of slopes between data pairs
- Not a test

Theil 1950, Sen 1968

Mann-Kendall Trend Test

non-parametric test

a special case of Kendall's tau test

- **Test for a monotonic trend**
  - not relying on an estimate of the trend itself (test only)
  - usually combined with Theil-Sen trend estimate
  - based on relative ranking of data (not on data itself)
- **Null hypothesis and alternative**
  - H_0: Time series values are independent, identically distributed
  - H_A: There is a monotonic (not necessarily linear) trend
- **Test statistic**

$$S = \sum_{i<k} \mathrm{sign}(x_k - x_i) \quad \text{with} \quad sign(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

Mann 1945, Kendall 1975

Mann-Kendall Trend Test

- **Null distribution**
  - For large samples (about n>8) S is normally distributed with

$$E(S) = 0, \quad \mathrm{var}(S) = \frac{n \cdot (n-1) \cdot (2n+5)}{18}$$

  - If the sample data is mutually independent (e.g. no serial corr.)
  - In practice use standardized test statistic Z and compare to Normal distribution

$$Z = \begin{cases} (S-1) \big/ \mathrm{var}(S)^{1/2} & S > 0 \\ 0 & S = 0 \\ (S+1) \big/ \mathrm{var}(S)^{1/2} & S < 0 \end{cases}$$

- **Remark**
  - Need correction of $\mathrm{var}(S)$ if there are many ties

see also Yue et al. 2002

Mann-Kendall Trend Test

- **Example**

$\beta = 0.087 \cdot m^3 \cdot s^{-1} / a$

$S = 387$
$Z = 1.96$

$q_N(0.025) = -1.96$
$q_N(0.975) = +1.96$

p-value: 0.05

➨ Trend is smaller and less significant than with linear regr.

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Accounting for Serial Correlation

Serial Correlation in Trend Tests

- **Serial correlation increases variance of trend estimates**

r_1=0.2 r_1=0.4 r_1=0.6 r_1=0.8

Trend estimates (Theil-Sen) in 1000 synthetic time series composed of a linear trend and an AR(1).
(Sample size: 100, Trend: 1.2*σ change over the full series, r_1: lag-1 autocorrelation.)

Yue et al. 2002

Serial Correlation in Trend Tests

- **Serial correlation affects the null distribution of a trend test**
- **False rejections (Type I errors) under H_0 are much more frequent than nominal rate. Test is more "liberal" than anticipated.**

False rejections of Mann-Kendall trend test for trendless AR(1) time series

significance level 5%

4 times the false rejections than nominal

Kulkarni & von Storch 1995; Yue et al. 2002

Serial Correlation in Trend Tests

- **In the presence of serial correlation, trends can be estimated less accurately.**
  - The larger the correlation, the larger the uncertainty
- **The Null-distribution becomes “broader” the larger the serial correlation.**
- **Conventional trend tests (without account of serial correlation) reject Null-Hypothesis too often.**
- **There is a tendency for “detecting” a trend even when there is none.**

Accounting for Serial Correlation

- **Pre-whitening:**
  - Idea: “wash-out” serial correlation from the data before testing
  - *Pre-whitened* data series:

$$x'_i = \left(x_i - r_1 \cdot x_{i-1}\right) / \left(1 - r_1\right)$$

r_1: lag-1 autocorrelation

  - If x_i is a combination of a trend + AR(1) then the pre-whitened series x'_i …

    … has the same trend as x_i ,

    … has serially uncorrelated residuals,

    … has a larger residual variance than the underlying AR(1).

the self-acting cleaning agent

von Storch 1995; Wang & Swail 2001, Yue et al. 2002, Zhang & Zwiers 2004

Derivation

- **See e.g. Wang and Swail 2001 (Appendix A)**

$$x_t = \alpha + \beta t + q_t \qquad q_t = r_1 \cdot q_{t-1} + \varepsilon_t \qquad AR(1)$$

$$x_t - r_1 \cdot x_{t-1} = \alpha - \alpha \cdot r_1 + \underbrace{\beta t - r_1 \beta (t-1)}_{r_1\beta + \beta t(1-r_1)} + \underbrace{q_t - r_1 q_{t-1}}_{\varepsilon_t}$$

$$= \alpha(1-r_1) + r_1 \cdot \beta + t \cdot \beta(1-r_1) + \varepsilon_t$$

$$x_t' = \alpha' + \beta \cdot t + \frac{\varepsilon_t}{(1-r_1)} \qquad \alpha' = \alpha + \beta \frac{r_1}{(1-r_1)}$$

$$\mathrm{var}(resid(x_t')) = \frac{\mathrm{var}(\varepsilon_t)}{(1-r_1)^2} \qquad \mathrm{var}(q_t) = \frac{\mathrm{var}(\varepsilon_t)}{\left(1-r_1^2\right)}$$

$$\mathrm{var}(resid(x_t')) = \mathrm{var}(q_t)\frac{(1+r_1)}{(1-r_1)} > \mathrm{var}(q_t)$$

Increase in variance

Pre-Whitening Example

by factor 4 !

original: r_1 = 0.75

Side-Effect of Pre-Whitening

- **Pre-whitening "washes-out" part of the trend**

r_1 estimated from AR(1) with trend:

➨ r_1 is notoriously overestimated in time series with trend.

Trend estimated after prewhitening:

➨ True trend is underestimated in pre-whitened time series.

Monte Carlo sims with n=100, linear trend, AR(1), Theil-Sen trend estimator.

Yue et al. 2002

Recursive Estimation

- **Recursive pre-whitening & detrending**

Wang & Swail 2001, Zhang & Zwiers 2004

Example

- **River Rhine: Mean annual runoff (Mainz)**

	Original	Results from recursive pre-whiteneding
Theil-Sen Trend:	2.9 $m^3\ s^{-1}\ a^{-1}$	3.3 $m^3\ s^{-1}\ a^{-1}$
Mann-Kendall Test (p-value):	0.10	0.17
Lag-1 autocorrelation:	0.23	0.20

Serial Correlation in Trend Tests

- **False rejections with prewhitening + Mann-Kendall**

significance level 5%

- False rejections close to nominal also with AR(1)

Kulkarni & von Storch 1995; Yue et al. 2002

Application

- **Streamflow trends at undisturbed. rivers in CH**

Mean seasonal streamflow

Quantiles of daily streamflow

Fraction of rivers showing statistically significant (α=0.1) trends. Using the Mann-Kendall trend test with prewhitening for serial correlation. Black bars for increasing trends, empty bars for decreasing trends.

Trend in maximum winter streamflow

Winter

Maximum winter streamflow trend for individual river catchments. Upward triangle for significant upward trend, dots for non-significant trends.

Birsan et al. 2005

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Trend Analysis for Count Data

Trend Analysis and Count Data

- **Event counts**
  - # number of frost days,
  - # of heat days (T_{max} > 30°C)

- **Linear regression ?**
  - Discrete and bounded data
  - Possibly neg. predictions
  - Skewed, resids not normal
  - Variable variance

- **Mann-Kendall ?**
  - Many ties, ranks unambiguous

# of winter days with rain > 12.5 mm
(3 events on average per winter)

Logistic Regression

parametric test

- **Based on binomial distribution**
  - $e(t)$: # events at t, a realisation of binomial random variable $E(t)$

$$E(t) \sim B(m, \pi(t))$$

  - m: max. possible counts (e.g. 90 winter days)
  - π: event probab. Time dependent.

- **Transformed linear time dependence**

$$\eta(\pi(t)) = \beta \cdot t + \alpha \qquad \eta(.) : \text{the link function}$$

$$\eta(x) = \text{logit}(x) = \log\left(\frac{x}{1-x}\right) \qquad x \in]0,1[\;\Rightarrow\; \eta(x) \in]-\infty, +\infty[$$

  - A random model for the event counts with parameters α, β.

McCullagh & Nelder 1989, see also Frei & Schär 2001

Generalised Linear Models

- **A broad class of regression models**
  - Including classical linear regression and logistic regression
- **Transformations of distribution parameters θ are linear functions of predictors x**

$$Y(x) \sim D(\theta(x), \sigma) \qquad \eta(\theta) = \alpha + \beta \cdot x$$

- **Suitable form of transformation η (link function) derives from theory**
- **E.g. linear regression**

$$Y(x) \sim N(\mu(x), \sigma) \qquad \eta(\mu) = \alpha + \beta \cdot x, \quad \eta(u) := u$$

McCullagh & Nelder 1989

Estimation

- **Maximum Likelihood Method**
  - Log-likelihood function

Binomial Prob. Distr.

$$L\left(\{e(t), t = 1,..,n\}; \alpha, \beta\right) = \sum_{t}^{n} \log\left(B\left(e(t); m, \pi(t; \alpha, \beta)\right)\right)$$

observed counts at time t

with:

$$\pi(t; \alpha, \beta) = \exp(\alpha + \beta \cdot t) / (1 + \exp(\alpha + \beta \cdot t))$$

inverse of logit function.

  - Determine those (α,β) for which L is largest

`R: glm`

Testing

- **Significance of trend parameter β:**
  - Maximum-Likelihood procedure delivers confidence bounds for α, β. (Curvature of Likelihood funktion near maximum.)
  - "Statistically significant trend" if confidence interval of β is on either side of 0 (does not contain 0).

R: `glm`

Trend Magnitude

- **The Odds-ratio,**

$$\Theta = \frac{\pi(n)}{(1-\pi(n))} \Big/ \frac{\pi(1)}{(1-\pi(1))} = \exp(\beta(n-1))$$

n: end point of time series

  - a measure of trend magnitude

  - For small event probabilities (π << 1):

$$\Theta \approx \pi(n)/\pi(1)$$

the ratio of event probabilities between the end/beginning

Logistic Regression

- **Example**

Odds ratio Θ : 2.88

p-value: $7 \cdot 10^{-5}$

Overdispersion: 1.6

$m \cdot \pi(t)$

Assumptions

- **Implied assumptions**
  - Counts are strictly binomially distributed
  - Probability is the same with each toss in a group
  - No auto-correlation

- **Variable probabilities and auto-correlation …**
  - … lead to larger than nominal binomial variance (overdispersion),
  - … inflate the dispersion of the Null-Distribution,
  - … can be corrected in testing, taking account of overdispersion in Null Distribution.

McCullagh & Nelder 1989

Overdispersion

Example: # days with intense precip

Logistic Regression

- **Example**

Odds ratio Θ : 2.88

p-value: $7 \cdot 10^{-5}$

Overdispersion: 1.6

Data variance is larger than nominal Binomial variance by a factor of 1.6

$m \cdot \pi(t)$

Logistic Regression

- **Application: Heavy precipitation trends in CH**

Trend in the # of intense precipition events in winter (events above the long-term 90% quantile) between 1901-2000. Blue=increase, red=decrease, filled symbols: p-value < 0.05

Evolution of intense precipitation events in Northern Switzerland. Median of events at individual stations.

Schmidli & Frei 2005

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Additional Remarks

Fishing Expeditions

- **Want to be scientifically "successful"? Then you should:**
  1) Conduct many hypothesis tests,
  2) Publish the significant results only.

- **Of 100 independent tests, 5% (or whatever α) will be statistically significant BY CHANCE!**

The Mexican Hat

- **Is the "Mexican Hat" natural (H_0) or man made (H_A)?**

- **For *development* and *testing* of a hypothesis you need TWO independent datasets.**

- **Preselection of data tricks the result of a hypothesis test.**

The Mexican Hat
desert of Utah/Arizona

von Storch 1995