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# Inventory Control in a Multi-Supplier System

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**Abstract**

An enterprise network is analyzed from the viewpoint of an end-product manufacturer who receives customer orders and organises his production and supply policy so as to minimize the sum of his average holding cost and average stock-out cost. For each main component to be ordered, the producer has several possible suppliers. The arrivals of customers' orders are random and delivery times from suppliers are also supposed random. This supply system is represented as a queuing network and the producer uses a base-stock inventory control policy that keeps constant the position inventory level (current inventory level + pending replenishment orders). The decision variables are the reference position inventory level and the percentages of orders sent to the different suppliers. In the queuing network model, the percentages of orders are implemented as Bernoulli branching parameters. A close-form expression of the expected cost criterion is obtained as a complex non-linear function of decision variables. A decomposed approach is proposed for solving the optimization problem in an approximate manner. The quality of the approximate solution is evaluated by comparison to the exact solution, which can be computed numerically in some simple cases, in particular in the two-supplier case. Numerical applications show the important economic advantage for the producer of sending orders to several suppliers rather than to a single one.

**Keywords:** Inventory Control, Supply Chain, Stochastic Models

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## 1. Introduction

A major difficulty in supply chain organisation and management is to conciliate global efficiency with local autonomy. When considering a network of cooperating enterprises, a basic objective is to organize integration in a non-compulsory manner, so as to maintain the autonomy of partners. A possible approach for combining integration and autonomy is through partly automated negotiation processes (Jennings et al, 2001, Besembel et al., 2002). Once the negotiation has started, parameters can be updated and other criteria can enter into play, such as costs (fixed and variable ordering costs), quality and non-formalized preference. In such a way, the system complexity inherently attached to supply chain organization, can be managed through negotiation between the main actors, each one of them basing his decision upon a local optimization process. Such a scheme seems appealing by preserving the autonomy and decision optimization among the partners of an Enterprise Network. However, it has been shown to be globally sub-optimal (see e.g. Cachon and Zypkin, 1999), by driving the system to a Nash equilibrium which can globally perform very poorly with respect to the minimal total cost criterion. Several corrective actions have been proposed to compensate for this bias. They mainly consist in sharing risks and costs among partners and this can be implemented through contracts modifying local criteria in a globally more efficient manner (Cachon and Lariviere, 2001, Chen et al., 2001).

Another well-known factor of inefficiency in supply chains is the so-called “bullwhip effect”, which tends to propagate and amplify disturbances upward along the supply chain. A supply chain generally involves several sources of disturbances, and coordination of product flows is fragile since variations in external supply and demand may be amplified through interconnections between partners. Some typical causes for such amplification are capacity limitations and the use of different batch sizes between partners (Lee and Billington, 1992).

The study analyzes one of the basic elements of a supply chain: the operational relationships between an end-producer and his direct suppliers. A simple queuing model is constructed, based on the assumptions of a Poisson external demand for end-products, instantaneous delivery to the customer from the producer's stock and an exponentially distributed service time for each supplier. Only one basic component of the end-product is considered and suppliers are supposed equivalent in terms of quality and cost. They only differ by their average service time. In spite of its simplicity, such a model grasps the main issues for the producer: should he use only one supplier, the best one in terms of a relevant performance index, or should he dispatch his orders between different suppliers? In the latter case, what suppliers should be selected and for what percentage of the demand?

The potential usefulness of the model for the supplier is in the a-priori determination of his "optimal" inventory level and of the volumes (or frequency) of his orders to suppliers, based on a-priori evaluation of their average delivery time. In practice, this a-priori knowledge can be considered as a starting point in the negotiation process that will be undertaken with the suppliers. By assuming unitary demands and orders and memoryless arrival, dispatching and service processes, we get rid of the bullwhip effect in the model, to concentrate on the mean performance analysis. From the literature on the bullwhip effect, it is assumed that it can be treated separately, through a more detailed model, either through synchronous scheduling (Cachon, 1999) or/and through an adequate choice of batch sizes (Riddalls and Bennett, 2001).

To optimize the producer's inventory level and the order dispatching proportions, it is essential to combine the effects of random fluctuations on demand flows, and delays of deliveries from suppliers. Random demands have often been considered in the existing models of inventory control, specially the ones based on the "newsvendor" paradigm (Arrow et al., 1951, Porteus, 1990). On the contrary, random delays in part deliveries have not often been integrated in models explicitly. An exception is the work of Dolgui and Louly (2001), in which several suppliers with random delivery delays are considered. In their work, the different suppliers provide different parts to be assembled by the producers. Then, there is interdependency between the inventory positions of the end product and of its components. However, the inventory positions of the different components can be independently controlled through the information and ordering system. On the contrary, the case of a centralized inventory of a single component does not offer the same possibility of decomposition.

In this paper, a centralized inventory control model is constructed to combine supply and demand randomness. The objective of the producer is to minimize his average cost by

constructing an ordering policy, defined by an optimal reference inventory position and a rule for selecting the supplier of each inventory replenishment order.

Section 2 formulates the optimal inventory and ordering problem for one producer and several suppliers. Then section 3 solves optimally the order dispatching problem in the particular "make – to – order case, and an approximate resolution technique is presented in section 4 for the make – to – stock case. The performance of the approximate solution is comparatively evaluated on simple examples in section 5. Finally, some conclusions and perspectives are presented.

## 2. The optimal inventory and ordering policy

The current inventory level of the product considered at time t is denoted *I(t)*. It is defined as the difference between the on-hand inventory and the amount of backorders. In general, an ordering decision should not be based only on the inventory level. One should also consider the number of replenishment orders which have been placed earlier and not yet been delivered, denoted *u(t)*. The global state of the system can then be characterized by the inventory position, denoted P(t) defined by:

$$P(t) = I(t) + u(t) \tag{1}$$

Depending on the information system available, an inventory position may be controlled at any time through a *continuous review* policy, or at periodic times through a *periodic review* policy. Then, the control policy determines when and how much to order. Different control policies may be applied, within the limits of the legal agreements between producer and supplier.

One of the most popular continuous review policy is the (s,S) policy, in which s stands for the inventory position order point and S for the inventory position replenishment level. The base-stock policy can be seen as a variant of the (s,S) policy, for which an order is placed whenever a demand comes, so as to permanently maintain the inventory position S. This type of a policy has been shown to be optimal under constant average demand rates or unitary demands with independent identically distributed (i.i.d.) arrival dates, whenever the cost criterion only depends on the inventory position (Axsater, 2000). Moreover, under unitary demands with (i.i.d.) arrival dates, the optimal base stock policy reduces to the policy (s,S) with s = S-1. This policy is denoted the *reference inventory policy*. It will be studied in the sequel, in the multi-supplier case.

In the "make-to stock" context, when an order comes to the producer, it is immediately satisfied if its amount is available in the stock. If not, it has to wait until the inventory has been sufficiently replenished by the arrival of products from suppliers. In both cases, an order is placed from the producer to the supplier whenever a demand comes and has the same amount (1 in the unitary case). As shown in (Bollon et al., 2000), such a base stock control policy can also be interpreted as a Kanban mechanism After an initial inventory replenishment stage, the reference inventory policy maintains constant the inventory position of the producer:

$$P(t) = S \quad \forall t \geq t_0 \tag{2}$$

If the random processes of demand arrivals and supply deliveries are stationary, then, under a stationary *(S-1,S)* base stock policy, the system (producer + suppliers) reaches stationary conditions characterized by stationary probabilities of the number of orders placed by the producer and not yet delivered by the suppliers. In the sequel, these probabilities will be computed in the case of exponential distributions of demand arrivals and supply lead times.

From the producer viewpoint, the cost function to be minimized is the sum of the average holding cost and the average stock-out cost. Consider the following notations:

$I$ is the random variable representing the producer inventory level in stationary conditions

$u$ is the random variable representing the number of uncompleted orders from the producer to the supplier in ergodic conditions

$h$ is the unit holding cost,

$b$ is the unit stock-out cost.

In stationary conditions under the *(S-1,S)* base stock policy, random variables $I$ and $w$ are related by the following equality, derived from (1) and (2):

$$I=S-u \tag{3}$$

Using notation $(x)^+$ for $max(x,0)$ and $(x)^-$ for $max(-x,0)$, the average cost criterion takes the form:

$$C(S) = E\,[\,h\,(I)^+ + b\,(I)^-\,]\,. \tag{4}$$

Demands are assumed unitary. They enter the system as a Poisson process with rate $\lambda > 0$.

When a unitary demand arrives at time $t$, the producer serves it immediately if $I(t)\geq 0$. He waits if $I(t)<0$. In both cases, he applies the *(S-1,S)* ordering policy by sending a corresponding order to supplier $i$ with probability $\alpha_i$ satisfying $0 \leq \alpha_i \leq 1$ and $\sum_{i=1}^{N} \alpha_i = 1$. Such

unitary orders can be seen as a limit case that maximizes the information efficiency of order transfer from the producer to the suppliers and minimizes the "bullwhip" effect by avoiding transmission distortions due to differences in batch policies. Each supplier is supposed to have an exponential service rate and treats the requests in the FIFO (First In First Out) order. Let $\mu_i$ denote the mean service rate of supplier $i = 1,...,N$, with $\mu_i \neq \mu_j$ for $j \in (1,...,i-1,i+1,...,N)$.
Such an order dispatching policy from the producer to the supplier is known as a Bernoulli splitting process. A well known property of this process is $\{N_i(t), t \geq 0\}$ $i = 1,...,N$ are Poisson processes with rates $p_i\lambda$. Moreover, these processes are mutually independent.The proof of this property can be found, in particular, in (Ross, 2000). As a consequence, each supplier can be modelled as an M/M/1 queue with arrival rate $\alpha_i\lambda$ and service rate $\mu_i$. The set of suppliers is represented as a network of N independent M/M/1 queues in parallel. The probability of having $k_i$ orders in queue $i$ is given by:

$$P(k_i) = \Pr\{\, k_i \text{ orders in } i \,\} = \left(\frac{\alpha_i \lambda}{\mu_i}\right)^{k_i} \left(1 - \frac{\alpha_i \lambda}{\mu_i}\right) \tag{5}$$

The necessary and sufficient condition for stability of queue i is $\alpha_i\, \rho_i < 1$, with $\rho_i = \lambda / \mu_i$. The probability for the network of queues to be in state $K = \{k_1, k_2, ..., k_i, ..., k_N\}$ is given by the product-form expression (Baskett *et al*, 1975):

$$P(k_1,...,k_N) = \prod_{i=1}^{N} (\alpha_i \rho_i)^k (1 - \alpha_i \rho_i) \tag{6}$$

In the order dispatching problem, Bernoulli parameters $\alpha_1$, $\alpha_2$… $\alpha_N$ are decision variables. Their optimal values express the optimal assignment probabilities in steady state. The considered optimization criterion is the sum of the mean holding cost and the mean stock-out cost per time unit. The objective of the study is to compute the optimal Bernoulli parameters and the optimal base stock value, $S^*$ minimizing the average cost criterion.
In the case of N suppliers, the number of orders not yet delivered to the producer is equal to the total number of orders in the open queuing network composed by the queues of orders coming from the producer. Let $K_i$ denote the number of orders sitting in the ith supplier queue. $P(k_i) = Pr\{K_i = k_i\}$ is defined by equation (5). Then, the probability of having w orders waiting in the suppliers queue is given by

$$P_w = \Pr\{\, K_1 + K_2 + ... + K_N = w \,\}$$

Probability $P_w$ can be obtained by composition of the probabilities related to the N queues. Such a composition can be computed from the probability generating function. Assuming

$\alpha_i \rho_i \neq \alpha_j \rho_j \ \forall i \neq j$, the product form (21) can be transformed into a sum as in (Kleinrock, 1975). Then one obtains (Arda and Hennet, 2003)

$$P_w = H(N) \times \sum_{i=1}^{N} \left( \left( \prod_{\substack{j=1 \\ j \neq i}}^{N} b_{ij} \right) \times \alpha_i^{N+w-1} \rho_i^{N+w-1} \right) \text{ with } H(N) = \prod_{i=1}^{N} (1 - \alpha_i \rho_i), b_{kj} = \frac{1}{\alpha_k \rho_k - \alpha_j \rho_j} . \quad (7)$$

The mean value of the number of pending orders is denoted Z, with

$$Z = E[u] = \sum_{w=0}^{\infty} w P_w \ . \quad (8)$$

Criterion (4) can be re-written:

$$C(S) = (h+b) \sum_{w=0}^{S} (S-w) P_w + b\,(Z - S) \quad (9)$$

and the following expression is obtained:

$$C(S, \alpha_1, \alpha_2, ..., \alpha_N) = (h+b) \times H(N) \times$$
$$\sum_{i=1}^{N} \left( \left( \prod_{\substack{j=1 \\ j \neq i}}^{N} b_{ij} \right) \times \left( \frac{S \alpha_i^{N-1} \rho_i^{N-1}}{1 - \alpha_i \rho_i} - \frac{\alpha_i^N \rho_i^N (1 - \alpha_i^S \rho_i^S)}{(1 - \alpha_i \rho_i)^2} \right) \right) + b \left( \sum_{i=1}^{N} \frac{\alpha_i \rho_i}{1 - \alpha_i \rho_i} - S \right). \quad (10)$$

Convexity of criterion (10) with respect to variables $\alpha_1, ..., \alpha_N, S$ is not guaranteed in general. Therefore, minimization of criterion (10) subject to constraints $0 \leq \alpha_i \leq 1$ for $i=1, ..., N$ and $\sum_{i=1}^{N} \alpha_i = 1$ appears to be a hard optimization problem.

## 3. The optimal solution in the MTO case

In the "Make-to Order" case, the base stock level is supposed equal to zero. Then, the cost function reduces to the backorder cost :

$$C(\alpha_1, \alpha_2, ..., \alpha_N) = bZ = b \sum_{i=1}^{N} \frac{\alpha_i \lambda}{\mu_i - \alpha_i \lambda} \quad (11)$$

Minimizing the backorder cost (11) is equivalent to minimizing the number of unsatisfied orders or equivalently, from Little formula, minimizing the expected waiting time E[T] = Z / λ. Without loss of generality, the N suppliers can be ranked in the decreasing order of their service rate : $\mu_1 > \mu_2 > ... > \mu_N > 0$. The problem constraints are based on the following conditions:

- Bernoulli parameters should be feasible. This condition requires the following constraints:

$$0 \le \alpha_i \le 1 \qquad i = 1,..., N \tag{12}$$

$$\sum_{i=1}^{N} \alpha_i = 1 \tag{13}$$

- Stability of the queuing network requires the following conditions :

$$\frac{\alpha_i \lambda}{\mu_i} < 1 \qquad i = 1,..., N \tag{14}$$

- Moreover, a necessary and sufficient condition for the existence of a set of Bernoulli parameters, ($\alpha_i$ ; i = 1,…,N) satisfying constraints (12), (13), and (14) is :

$$\lambda < \sum_{i=1}^{N} \mu_i \tag{15}$$

Suppose that condition (15) is satisfied by the problem data. Then, the "Make-to Order" optimization problem, denoted **problem (P1)** takes the following form :

$$\underset{\alpha_1,...,\alpha_N}{minimize} \sum_{i=1}^{N} \frac{\alpha_i}{\mu_i - \alpha_i \lambda} \tag{16}$$

under constraint (13), and constraint (17) which replaces (12) and (14) :

$$0 \le \alpha_i \le \min(1, \frac{\mu_i}{\lambda}) \qquad i = 1,..., N \tag{17}$$

All the constraints are linear and in the feasible domain, criterion E[T] is convex:

$$\frac{d^2 E[T]}{d\alpha_i^2} = \frac{d}{d\alpha_i}\left(\frac{\mu_i}{(\mu_i - \alpha_i\lambda)^2}\right) = \frac{2\mu_i\lambda(\mu_i - \alpha_i\lambda)}{(\mu_i - \alpha_i\lambda)^4} = \frac{2\mu_i\lambda}{(\mu_i - \alpha_i\lambda)^3} .$$

Non negativity of $\frac{d^2 E[T]}{d\alpha_i^2}$ is always guaranteed under constraint (14). Therefore, problem (P1) is convex and has a unique minimum defined by the first order optimality conditions:

$\frac{d\, E[T]}{d\alpha_i} = 0$ for i=1,…,N under constraints (13), and (17).

### *3.1 Resolution of a relaxed problem*

Consider now the case when the optimal solution of the problem defined by (16) and (13) naturally satisfies condition (17). Then, this solution is optimal for problem (P1). The Lagrangean of the relaxed problem can be written :

$$L = \left( \sum_{i=1}^{N} \frac{\alpha_i}{\mu_i - \alpha_i \lambda} \right) - p \left( \sum_{i=1}^{N} \alpha_i - 1 \right)$$

with p the Lagrange parameter associated with the equality constraint : $\sum_{i=1}^{N} \alpha_i = 1$. Let $\alpha_i^*$ be the optimal value of the Bernoulli parameter $\alpha_i$ for i = 1,…,N. Then, the optimal solution of the relaxed problem satisfies the following set of conditions:

$$\frac{d\,L}{d\alpha_i} = \frac{\mu_i}{(\mu_i - \alpha_i^* \lambda)^2} - p = 0 \qquad i = 1,..., N \tag{18}$$

$$\sum_{i=1}^{N} \alpha_i^* = 1 \tag{19}$$

For any pair ($\alpha_i^*$, $\alpha_j^*$), condition (18) can be re-written :

$$\mu_j - \alpha_j^* \lambda = \frac{\sqrt{\mu_j}\ (\mu_i - \alpha_i^* \lambda)}{\sqrt{\mu_i}} \tag{20}$$

By summing over j both terms of equation (20), one obtains:

$$\sum_{j=1}^{N} (\mu_j - \alpha_j^* \lambda) = \sum_{j=1}^{N} \frac{\sqrt{\mu_j}\ (\mu_i - \alpha_i^* \lambda)}{\sqrt{\mu_i}}$$

Under constraint $\sum_{i=1}^{N} \alpha_i^* = 1$, this equation becomes : $\sum_{j=1}^{N} \mu_j - \lambda = \frac{\mu_i - \alpha_i^* \lambda}{\sqrt{\mu_i}} \sum_{j=1}^{N} \sqrt{\mu_j}$ and thus, the following result is obtained.

**Property 1 :**

*The optimal values of Bernoulli parameters with respect to the relaxed problem, are defined by the following expressions:*

$$\alpha_i^* = \frac{1}{\lambda} (\mu_i - \tau_N \sqrt{\mu_i}) \qquad i = 1,..., N \tag{21}$$

*where $\tau_N$ is defined by :*

$$\tau_N = \frac{\sum_{j=1}^{N} \mu_j - \lambda}{\sum_{j=1}^{N} \sqrt{\mu_j}} \tag{22}$$

If the optimal values ($\alpha_i$*) satisfy constraints (17), then the minimum of the relaxed problem is feasible and therefore optimal for problem (P1). The feasibility condition is re-written:

$$0 \le \frac{1}{\lambda} (\mu_i - \tau_N \sqrt{\mu_i}) \le \min(1, \frac{\mu_i}{\lambda}) \qquad i = 1,..., N \tag{23}$$

From condition (13), $\tau_N > 0$. Therefore, inequalities (23) can be replaced by :

$$0 \le \mu_i - \tau_N \sqrt{\mu_i} \le \lambda \qquad i = 1,..., N \tag{24}$$

meaning $0 \le \alpha_i^* \le 1$. The left-side inequality can be re written :

$$\mu_i \ge \tau_N^2 \qquad i = 1,..., N \tag{25}$$

And, using $\sum_{i=1}^{N} \alpha_i^* = 1$, the right-side inequality becomes satisfied.

### *3.2 The restricted choice problem*

From the ranking of service rates in the decreasing order $\mu_1 > \mu_2 > ... > \mu_N > 0$, if. inequality (25) is not satisfied for $i_0$, with $1 \le i_0 \le N$, then, it is also violated for i = $i_0$ +1,…,N. In this case, the "restrictive choice problem" is obtained by imposing $\alpha_i^* = 0$ for i = $i_0$,…,N To show the relevance of the restricted choice problem, the following parameter is defined for m=1,…,N+1, under the convention $\mu_{N+1} = 0$:

$$\tau_m = \frac{\sum_{j=1}^{m} \mu_j - \lambda}{\sum_{j=1}^{m} \sqrt{\mu_j}} \tag{26}$$

The evolution of $\tau_m$ satisfies the following properties.

**Property 2**

*For positive values of parameters $\tau_m$ et $\tau_{m+1}$ ($m \le N-1$), the evolution of $\tau_m$ satisfies the following rules :*

$$(1) \begin{cases} \tau_{m+1} < \tau_m \\ \Leftrightarrow \mu_{m+1} < \tau_m^2 \\ \Leftrightarrow \mu_{m+1} < \tau_{m+1}^2 \end{cases} \quad (2) \begin{cases} \tau_{m+1} = \tau_m \\ \Leftrightarrow \mu_{m+1} = \tau_m^2 \\ \Leftrightarrow \mu_{m+1} = \tau_{m+1}^2 \end{cases} \quad (3) \begin{cases} \tau_{m+1} > \tau_m \\ \Leftrightarrow \mu_{m+1} > \tau_m^2 \\ \Leftrightarrow \mu_{m+1} > \tau_{m+1}^2 \end{cases}$$

**Proof**

From (26), one obtains

$$\tau_m \sum_{i=1}^{m+1} \sqrt{\mu_i} - \tau_m \sqrt{\mu_{m+1}} = \tau_m \sum_{i=1}^{m} \sqrt{\mu_i} = \sum_{i=1}^{m+1} \mu_i - \mu_{m+1} - \lambda \tag{27}$$

$$\tau_{m+1} \sum_{i=1}^{m} \sqrt{\mu_i} + \tau_{m+1} \sqrt{\mu_{m+1}} = \tau_{m+1} \sum_{i=1}^{m+1} \sqrt{\mu_i} = \sum_{i=1}^{m} \mu_i + \mu_{m+1} - \lambda \tag{28}$$

this implies :

$$\mu_{m+1} - \tau_m \sqrt{\mu_{m+1}} = (\tau_{m+1} - \tau_m) \sum_{i=1}^{m+1} \sqrt{\mu_i} \tag{29}$$

$$\mu_{m+1} - \tau_{m+1} \sqrt{\mu_{m+1}} = (\tau_{m+1} - \tau_m) \sum_{i=1}^{m} \sqrt{\mu_i} \tag{30}$$

The rules of property 2 directly derive from these two equalities. □

**Property 3 :**

*Parameter $\tau_m$ increases with m for $1 \leq m \leq m^*$. Then, parameter $\tau_m$ monotonously decreases with m for $m^* < m \leq N$. The maximal value of parameter $\tau_m$ is obtained for $m^*$ ($1 \leq m^* \leq N$), which is the unique index satisfying:*

$$\sum_{i=1}^{m^*} \mu_i > \lambda \quad \text{and} \quad \mu_{m^*+1} \leq \tau_{m^*}^2 < \mu_{m^*} \tag{31}$$

**Proof :**

The proof is presented in two parts.

i) Existence of the index $m^*$ :

For any set of parameters ($\lambda$, $\mu_1$,…, $\mu_N$), condition (26) implies $\tau_N > 0$. Let n be the smallest index satisfying $\sum_{i=1}^{n} \mu_i > \lambda$ . Replacing m + 1 by *n* in equation (31), one obtains:

$$\mu_n - \tau_n \sqrt{\mu_n} = (\tau_n - \tau_{n-1}) \sum_{i=1}^{n-1} \sqrt{\mu_i} \tag{32}$$

From $\tau_n > 0$ and $\tau_{n-1} \leq 0$, equation (33) implies $\mu_n > \tau_n^2$ . If $\mu_{n+1} \leq \tau_n^2$ , then $m^* = n$. If not, relation $\mu_{n+1} > \tau_n^2$ implies $\mu_{n+1} > \tau_{n+1}^2$ and $\tau_{n+1} > \tau_n$ by the third rule of Property 2. And so, the process is iterated for m = n + 2,…,N.

Then, starting from $\mu_{N+1} = 0$, we obtain $\mu_{N+1} \leq \tau_N^2$ . Then, if $\mu_N > \tau_N^2$, $m^* = N$. If not, relation $\mu_N \leq \tau_N^2$ implies $\mu_N \leq \tau_{N-1}^2$ et $\tau_N \leq \tau_{N-1}$ from the first rule of Property 2. Thus, there exists a unique index $m^*$, with $1 \leq m^* \leq N$, that satisfies relations (31). Note that in the case n = N, equation (32) implies $\mu_N > \tau_{N+1}^2$ and thus $m^* = N$.

ii) The evolution of parameter $\tau_m$ follows property 3.

For indices $m=m^*,\ldots,N$, relation $\sum_{i=1}^{m}\mu_i > \sum_{i=1}^{m^*}\mu_i > \lambda$ implies $\tau_m > 0$.Then, from $\mu_{m^*+1} \le \tau_{m^*}^2$, we can derive from Property 2, $\tau_{m^*+1} \le \tau_{m^*}$ and $\mu_{m^*+1} \le \tau_{m^*+1}^2$ Therefore, relation $\mu_{m^*+2} < \mu_{m^*+1} \le \tau_{m^*+1}^2$ implies $\tau_{m^*+2} < \tau_{m^*+1}$. Applying the same reasoning for $m = m^*+3,\ldots,N$, shows that parameter $\tau_m$ monotonously decreases with m for $m^* < m \le N$.

According to equations (29) and (30), parameter $\tau_m$ increases with m for $1 \le m \le n-1$, since $\mu_i > 0$ for $i = 1,\ldots,N$. Then, knowing that $\tau_m > 0$ for $m = n,\ldots,m^*$ and that $\mu_{m^*} > \tau_{m^*}^2$, one can write $\tau_{m^*} > \tau_{m^*-1}$ and $\mu_{m^*} > \tau_{m^*-1}^2$ from Property 2. Consequently, relation $\mu_{m^*-1} > \mu_{m^*} > \tau_{m^*-1}^2$ implies $\tau_{m^*-1} > \tau_{m^*-2}$. The same reasoning can then be applied to $m = n,\ldots,m^*-3$. And thus, the value of parameter $\tau_m$ increases with m for $1 \le m \le m^*$ and finally, the maximal value of parameter $\tau_m$ is obtained for $m^*$. □

If condition (25) is not satisfied, the constrained problem can be solved using the following property :

**Property 4 :**

*Suppose that condition (13) is satisfied and consider the index m* ($1 \le m^* \le N$) which satisfies relations (31). Then, the optimal values of Bernoulli parameters are given by:*

$$\alpha_i^* = \begin{cases} \frac{1}{\lambda}(\mu_i - \tau_{m^*}\sqrt{\mu_i}) & \text{for } i = 1,\ldots,m^* \\ 0 & \text{for } i = m^*+1,\ldots,N \end{cases} \tag{33}$$

**Proof :**

(i) Feasibility of policy $\alpha^*(m^*)$ defined by property 4 :

The set of Bernoulli parameters ($\alpha_i^*$ ; $i = 1,\ldots,N$) defined by (33) satisfies constraint (13) :

$$\sum_{i=1}^{N}\alpha_i^* = \sum_{i=1}^{m^*}\frac{1}{\lambda}(\mu_i - \tau_{m^*}\sqrt{\mu_i}) = \frac{1}{\lambda}(\sum_{i=1}^{m^*}\mu_i - \tau_{m^*}\sum_{i=1}^{m^*}\sqrt{\mu_i}) = 1.$$

If $m^* = 1$, the set ($\alpha_1^* = 1, \alpha_i^* = 0$ for $i = 2,\ldots,N$) is feasible. If $m^* > 1$, since $\mu_i > \mu_{m^*}$ for $i = 1,\ldots m^*-1$ and $\mu_{m^*} > \tau_{m^*}^2$, then $\mu_i > \mu_{m^*} > \tau_{m^*}^2$ which implies $\mu_i > \tau_{m^*}\sqrt{\mu_i}$ for $i = 1,\ldots m^*$. From expression (33), it follows $\alpha_i^* > 0$ for $i = 1,\ldots m^*$. Moreover, the right side inequality of constraint (15) is satisfied since $\tau_{m^*} > 0$. Thus, property 4 defines a feasible policy.

(ii) Optimality of policy $\alpha^*(m^*)$ :

From property 4, the policy $\alpha^*(m^*)$ is optimal if $m^* = N$. For $m^* < N$, from the convexity of problem P1 with respect to parameters $\alpha_j$, for $j = 1,...,N,$, it suffices to show that the set of values ($\alpha_1^*, \alpha_2^*,..., \alpha_{m^*}^*,0,...,0$) is locally optimal. So, consider an increase $\partial\alpha_j > 0$ with $m^* < j \leq N$. from policy $\alpha^*(m^*)$. Then, constraint (13) implies a decrease of $\alpha_i^*$ for some $i = 1,\ldots,m^*$ ($\partial\alpha_i < 0$) under the following feasibility constraint:

$$\partial\alpha_j + \sum_{i=1}^{m^*} \partial\alpha_i = 0 \tag{34}$$

The criterion variation then takes the following form :

$$\partial E[T] = \left( \sum_{i=1}^{m^*} \frac{\alpha_i^* + \partial\alpha_i}{\mu_i - (\alpha_i^* + \partial\alpha_i)\lambda} - \frac{\alpha_i^*}{\mu_i - \alpha_i^*\lambda} \right) + \frac{\partial\alpha_j}{\mu_j - \partial\alpha_j\,\lambda} \tag{35}$$

or, equivalently,

$$\partial E[T] = \left( \sum_{i=1}^{m^*} \frac{\partial\alpha_i\ \mu_i}{(\mu_i - (\alpha_i^* + \partial\alpha_i)\lambda)(\mu_i - \alpha_i^*\lambda)} \right) + \frac{\partial\alpha_j}{\mu_j - \partial\alpha_j\,\lambda} \tag{36}$$

From relations $\partial\alpha_j > 0$, $\partial\alpha_i < 0$, and expression (33), the following inequality is obtained.

$$\partial E[T] > \left( \sum_{i=1}^{m^*} \frac{\partial\alpha_i\ \mu_i}{(\mu_i - \alpha_i^*\lambda)^2} \right) + \frac{\partial\alpha_j}{\mu_j} = \left( \sum_{i=1}^{m^*} \frac{\partial\alpha_i}{\tau_{m^*}^2} \right) + \frac{\partial\alpha_j}{\mu_j} = \frac{1}{\tau_{m^*}^2} \sum_{i=1}^{m^*} \partial\alpha_i + \frac{\partial\alpha_j}{\mu_j} \tag{37}$$

Then, using equation (34), $\sum_{i=1}^{m^*} \partial\alpha_i$ can be replaced by $-\partial\alpha_j$ in inequality (37). And from $\mu_j \leq \tau_{m^*}^2$ for $j = m^*-1,\ldots,N$, we obtain :

$$\partial E[T] > \partial\alpha_j \left( \frac{1}{\mu_j} - \frac{1}{\tau_{m^*}^2} \right) > 0 \tag{38}$$

Therefore, the policy defined by parameters (33) is optimal.

## 4. An approximate solution in the "make-to-stock" case

The "make-to-stock" case corresponds to the general case, including the "make-to-order" case, which can be characterized by a null base-stock level *(S=0)*. Due to the complexity of the cost function (10), it is proposed to decompose the problem into two parts. In the first part, Bernoulli parameters are the decision variables while the base stock level is supposed to take the zero value. These assumptions are the same as for Problem (P1). They correspond to the MTO (Make to Order) case solved at the preceding section. In the second part of the problem;

denoted (P2), the values of Bernoulli parameters are supposed given and the only decision variable to be determined is the base stock level. In this second part, the Bernoulli parameters values obtained in (P1) are used as input data for problem (P2) and the optimal value of the inventory capacity, *S**, is computed using the discrete version of the "newsvendor" problem.

### *4.1 Computation of the dispatching parameters*

Problem (P1) is solved as described in section 3. The value of m* is calculated by Property 3. Then, the reference values of Bernoulli parameters are directly computed by explicit expressions (33). The error derived from the application of this approximation scheme will be evaluated in section 5.

### *4.2 Determination of the base stock level*

From criterion expression (9), consider the incremental function $G(S) = C(S+1) - C(S)$. One obtains: $G(S) = (h+b)$ Prob $(u \le S)$ $-b$. The PDF $F(S)$ = Prob$\{ u \le S \}$ being a monotonous increasing function, so is G(S). Then, the value $S^*$ for which $C(S^*)$ is optimal satisfies:

$$\begin{cases} C(S^*) \le C(S^*-1) \Leftrightarrow G(S^*-1) \le 0 \\ C(S^*) < C(S^*+1) \Leftrightarrow G(S^*) > 0 . \end{cases} \tag{39}$$

Therefore, a necessary and sufficient condition for optimality is given by the condition:

$$\sum_{w=0}^{S^*-1} P_w \le \frac{b}{h+b} < \sum_{w=0}^{S^*} P_w . \tag{40}$$

For the order dispatching policy $\alpha^*(m^*)$, expression (7) of $P_w$ leads to evaluation the following quantity, from which the solution S* of Problem (P2) can be computed from (40):

$$\sum_{w=0}^{S} P_w = H(N) \times \sum_{i=1}^{m^*} \left( \left( \prod_{\substack{j=1 \\ j \ne i}}^{m^*} b_{ij} \right) \times \frac{\alpha_i^{m^*-1} \rho_i^{m^*-1} (1 - \alpha_i^{S+1} \rho_i^{S+1})}{1 - \alpha_i \rho_i} \right) .$$

## 5. Evaluation of the Approximate Method

The approximation scheme described in section 4 relies on two simplifications. The first one consists in replacing the global optimisation problem, with variables $\alpha_1,...,\alpha_N$ and $S$ by an independent problem (problem P1) in $\alpha_1,...,\alpha_N$, followed by a problem in $S$ (problem P2). The second simplification consists in solving problem (P1) for a value of $S$ ($S=0$) which is not, in general, the optimal one. It can be noted that with the value of $S$ imposed in problem

(P1), it is not possible to iterate the approximation scheme by updating the value of *S*. As a consequence, the quality of the approximate solution is not guaranteed and there is a possibility to identify some bias in the method. The approximate scheme has been evaluated in the particular case of one producer and two suppliers. In this case, the global optimal solution can be easily computed by exploration of the feasible domain (Arda and Hennet, 2003). Numerical evaluations reported on table 1, show an economic advantage for the producer of sending orders to several suppliers rather than to a single one, even when the second one is clearly less efficient than the first one. They also show that the approximation method is satisfactory with an average deviation of less than 8% from the optimum, but a strong tendency (more than 12%) to over-evaluate the dispatching parameters associated with the most efficient suppliers.

| | | $\alpha 1$ | $\alpha 2$ | S | Criterion |
|---|---|---|---|---|---|
| Example 1 $\mu 1$=1.25 $\mu 2$=0.5 | Optimal solution with 1 supplier | 1,000 | 0,000 | 30,000 | 30,957 |
| | Optimal solution with 2 suppliers | 0,740 | 0,260 | 15,000 | 14,494 |
| | Sub-optimal solution with 2 suppliers | 0,791 | 0,209 | 16,000 | 15,501 |
| Example 2 $\mu 1$=1.25 $\mu 2$=0.6 | Optimal solution with 1 supplier | 1,000 | 0,000 | 30,000 | 30,957 |
| | Optimal solution with 2 suppliers | 0,698 | 0,302 | 14,000 | 13,275 |
| | Sub-optimal solution with 2 suppliers | 0,748 | 0,252 | 14,000 | 13,969 |
| Example 3 $\mu 1$=1.25 $\mu 2$=0.7 | Optimal solution with 1 supplier | 1,000 | 0,000 | 30,000 | 30,957 |
| | Optimal solution with 2 suppliers | 0,660 | 0,340 | 13,000 | 12,290 |
| | Sub-optimal solution with 2 suppliers | 0,707 | 0,293 | 13,000 | 12,727 |
| Example 4 $\mu 1$=1.25 $\mu 2$=0.8 | Optimal solution with 1 supplier | 1,000 | 0,000 | 30,000 | 30,957 |
| | Optimal solution with 2 suppliers | 0,626 | 0,374 | 12,000 | 11,449 |
| | Sub-optimal solution with 2 suppliers | 0,667 | 0,333 | 12,000 | 11,726 |
| Example 5 $\mu 1$=1.25 $\mu 2$=0.9 | Optimal solution with 1 supplier | 1,000 | 0,000 | 30,000 | 30,957 |
| | Optimal solution with 2 suppliers | 0,596 | 0,404 | 11,000 | 10,738 |
| | Sub-optimal solution with 2 suppliers | 0,628 | 0,372 | 11,000 | 10,908 |
| Example 6 $\mu 1$=1.25 $\mu 2$=1 | Optimal solution with 1 supplier | 1,000 | 0,000 | 30,000 | 30,957 |
| | Optimal solution with 2 suppliers | 0,632 | 0,368 | 8,000 | 7,906 |
| | Sub-optimal solution with 2 suppliers | 0,667 | 0,333 | 8,000 | 8,049 |
| Example 7 $\mu 1$=2 $\mu 2$=0.5 | Optimal solution with 1 supplier | 1,000 | 0,000 | 9,000 | 9,955 |
| | Optimal solution with 2 suppliers | 0,845 | 0,155 | 9,000 | 8,614 |
| | Sub-optimal solution with 2 suppliers | 1,000 | 0,000 | 9,000 | 9,955 |
| Example 8 $\mu 1$=2 $\mu 2$=0.6 | Optimal solution with 1 supplier | 1,000 | 0,000 | 9,000 | 9,955 |
| | Optimal solution with 2 suppliers | 0,825 | 0,175 | 8,000 | 8,280 |
| | Sub-optimal solution with 2 suppliers | 0,966 | 0,034 | 9,000 | 9,438 |
| Example 9 $\mu 1$=2 $\mu 2$=0.7 | Optimal solution with 1 supplier | 1,000 | 0,000 | 9,000 | 9,955 |
| | Optimal solution with 2 suppliers | 0,790 | 0,210 | 8,000 | 7,992 |
| | Sub-optimal solution with 2 suppliers | 0,932 | 0,068 | 9,000 | 9,051 |
| Example 10 $\mu 1$=2 $\mu 2$=0.8 | Optimal solution with 1 supplier | 1,000 | 0,000 | 9,000 | 9,955 |
| | Optimal solution with 2 suppliers | 0,760 | 0,240 | 8,000 | 7,780 |
| | Sub-optimal solution with 2 suppliers | 0,897 | 0,103 | 8,000 | 8,673 |
| Example 11 $\mu 1$=2 $\mu 2$=0.9 | Optimal solution with 1 supplier | 1,000 | 0,000 | 9,000 | 9,955 |
| | Optimal solution with 2 suppliers | 0,730 | 0,270 | 8,000 | 7,627 |
| | Sub-optimal solution with 2 suppliers | 0,863 | 0,137 | 8,000 | 8,256 |
| Example 12 $\mu 1$=2 $\mu 2$=1 | Optimal solution with 1 supplier | 1,000 | 0,000 | 9,000 | 9,955 |
| | Optimal solution with 2 suppliers | 0,710 | 0,290 | 7,000 | 7,356 |
| | Sub-optimal solution with 2 suppliers | 0,828 | 0,172 | 8,000 | 7,953 |

*Table 1 Comparative Results*

## 6. Conclusions

Cooperation between the actors of a supply chain is a difficult problem due to the distributed nature of the system and the associated degrees of decisional autonomy of the actors.

Negotiation can be seen as a basic tool to combine autonomy and integration. However, at the present time, there is a lack of decision support tools for negotiation. In the particular case of a negotiation between one producer and N suppliers, the producer needs to have a clear vision of his own interest in terms of costs and delay. The study has shown that in the case of a random demand from customers and random delivery delays from suppliers, it is generally profitable to dispatch the orders between several suppliers rather than to direct all the replenishment orders toward a single one. More specifically, the addressed problem was to determine the percentages of orders to be directed toward each supplier and the base stock level. An approximate technique has been proposed to solve this problem. Even if the quality of this technique is satisfactory, an on-going research is devoted to its improvement.

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