Lecture 4: Amortization and Bond

Goals:

- Study Amortization with non-level interests and non-level payments.
- Generate formulas for amortization.
- Study Sinking-fund method of loan repayment.
- Introduce bond.

Suggested Textbook Readings: Chapter 3: §3.1 - §3.3 (except §3.2.1: Mortgage Loans in Canada, §3.2.2: Mortgage Loans in the US) and Chapter 4: §4.1

Practice Problems:

§3.1: 1-5, 9
§3.2: 2-12, 34
§3.3: 1-4, 6
§4.1: 1-4

The Amortization Method of Loan Repayment

Amortization Formulas:

- Balance: The original amount is equal to the present value of all payments

  $$L = OB_0 = K_1 v + K_2 v^2 + \cdots + K_n v^n$$

  The outstanding balance at time t should be the present value of all remaining payments.

  $$\begin{aligned} OB_t &= L(1+i)^t - K_1(1+i)^{t-1} - K_2(1+i)^{t-2} - \cdots - K_{t-1}(1+i) - K_t \\ &= K_{t+1} v + \cdots + K_n v^{n-t} \end{aligned}$$

- Interest: The interest at time t is the interest generated on the outstanding principal at time $t-1$.

  $$I_t = i \cdot OB_{t-1}$$

- Principal: the repaid principal is the payment minus the interest due.

  $$PR_t = K_t - I_t$$

It can happen that the payments or/and interest rates vary from one period to the next.

Example 1: (Example 3.2) *A loan of 1000 is repaid by 6 monthly payments, starting one month after the loan is made. The interest rate for the first three months is $i^{(12)} = 12\%$ and the interest rate for the next three months is $j^{(12)} = 6\%$. The first three payments are X each and the final three are $2X$ each. Construct the amortization schedule for this loan.*

Example 2: (Example 3.4) *A loan of 3000 at an effective quarterly interest rate of* $j = 0.02$ *is amortized by means of 12 quarterly payments, beginning one quarter after the loan is made. Each payment consists of a principal repayment of* 250 *plus interest due on the previous quarter's outstanding balance.*

(a) Construct the amortization schedule.

(b) **(Exercise 3.2.10)** *Find the total present value of the interest payments and principal payments separately.*

Repayment with level payments is the most common form of repaying a loan.

Amortization Formulas:

- Balance: The original amount is

$$L = Kv + Kv^2 + \cdots + Kv^n = Ka_{\overline{n}|i}$$

  The outstanding balance at time t is

$$OB_t = Kv + \cdots + Kv^{n-t} = Ka_{\overline{n}|i}(1+i)^t - Ka_{\overline{t}|i}(1+i)^t = Ka_{\overline{n-t}|i}$$

- Interest:

$$I_t = i \cdot Ka_{\overline{n-t+1}|i} = K(1 - v^{n-t+1})$$

- Principal:

$$PR_t = K - I_t = Kv^{n-t+1}$$

Example 3: *Smith takes out a* \$100,000 *mortgage to buy a house. The mortgage is repaid with monthly payments of* \$716.43*, starting one month after the mortgage begins, for 20 years at a nominal rate of* $i^{(12)} = 6\%$. *How much of the 200th payment is used to repay interest?*

The Sinking-Fund Method of Loan Repayment

The sinking fund method of repaying a loan makes deposit into a separate fund (called sinking-fund, often earning a lower interest rate than the loan), with periodic payments of interest only during the term of the loan, along with repayment of the full principal at the end of the term. The sinking fund method was frequently used to repay debt by governments in the 18th and 19th centuries, and is still often used by towns.

Sinking-fund method

The loan of L with effective interest i is repaid by the sinking-fund method. n level deposits are made to a sinking-fund earning effective interest j.

- Interest payment per period: $L \cdot i$
- Periodic deposit to the sinking-fund: $\dfrac{L}{s_{\overline{n}|j}}$
- The total periodic outlay: $L\left(i + \dfrac{1}{s_{\overline{n}|j}}\right)$

Sinking-fund method schedule

- Balance: $OB_t = L - \dfrac{L}{s_{\overline{n}|j}} \cdot s_{\overline{t}|j}$
- Interest paid in the t-th payment: $I_t = L\left[i - \dfrac{(1+j)^{t-1} - 1}{s_{\overline{n}|j}}\right]$
- Principal repaid in the t-th Period: $PR_t = \dfrac{L(1+j)^{t-1}}{s_{\overline{n}|j}}$

Example 4: (Example 3.6) *A loan of 100,000 is to be repaid by ten annual payments beginning one year after the loan is made. The lender wants annual payments of interest at a rate of* 10% *and repayment of the principal in a single lump sum at the end of 10 years. The borrower can accumulate the principal in a sinking fund earning an annual interest rate of* 8%. *Find the borrower's total annual outlay and compare this to the level annual payment required by the amortization method at* 10%.

Determination of Bond Prices

Governments can raise funds by borrowing, in the short term by issuing Treasury bills, and in the longer term by issuing coupon bonds.

Definition: A bond is an interest-bearing certificate of public (government) or private (corporate) indebtedness.

That is, a bond is a debt that usually requires periodic interest payments (called *coupons*) at a specified rate (called *coupon rate*) for a stated term (called *term of maturity*) and also requires the return of the principal (called *face value, par value or redemption amount*) at the end of the term (called *the maturity date*).

The face value and coupons of a bond are most often fixed, but the price of the bond may vary. The yield rate, i.e., the rate of return, on the bond varies with the price. Many of our calculations will involve determining the price of a bond from the yield rate, and vice versa.

Determination of bond prices

The following notation will be used to represent the various parameters associated with a bond:

- F – the face amount (par value) of the bond
- r – the coupon rate per coupon period (six months unless otherwise stated)
- C – the redemption amount on the bond (equal to the face value F unless otherwise stated)
- n – the number of coupon periods until maturity or the term of the bond
- j – the yield rate per coupon period

The purchase price of the bond is determined as the present value, on the purchase date. The bond price is

$$P = Cv_j^n + Fra_{\overline{n}|j} = C + (Fr - Cj)a_{\overline{n}|j}$$

If $C = F$, then

$$P = Fv_j^n + Fra_{\overline{n}|j} = F + F(r - j)a_{\overline{n}|j}$$

Let $K = Fv_j^n$, then $P = K + \frac{r}{j}(F - K)$. This is known as Makeham's formula.

Example 5: (Example 4.1) *A* 10% *bond with semiannual coupons has a face value of 1000 and was issued on June 18, 1990. The maturity date is June 18, 2010. If the yield rate is* $i^{(2)} = 5\%$*,*

(a) find the price of the bond on its issue date.

(b) Find the price of the bond on June 18, 2000, just after the coupon is paid.