PLANNING AND SCHEDULING PROBLEMS IN MANUFACTURING SYSTEMS WITH HIGH DEGREE OF RESOURCE DEGRADATION

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To my beloved grand parents and parents

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SUMMARY

Mortality is the fundamental property of all matter. Various types of equipment used in manufacturing industry are no exception. A machine, tool-group or piece of equipment, jointly referred to a 'resource' may deteriorate in many possible ways. Depending on the form of degradation, preventive or corrective measures are employed. The preventive measures such as machine maintenance, equipment/ process monitoring etc. help keep the equipment functioning smoothly while corrective measures such as replacement, repair etc. aim at minimizing disruptions in production due to resource failure. In order to fulfill the production requirements, it is imperative that the resource management decisions (including preventive and corrective decisions) be taken in an optimal manner. More often than not, the resource management decisions are affected by other production related decisions like production planning and scheduling, job inspection, etc. The central theme of this thesis is to address the different ways in which manufacturing resources deteriorate and develop optimization models for resource management in conjunction with other production related decisions.

Oftentimes, specialized equipment, used in relatively large quantities on the production floor are amenable to break frequently with use. The best example is a steel, aluminum or plastic mold (also called a die) used for the manufacturing of a variety of plastic goods, china, art work, machinery, electronics and building materials. After many uses and cleaning, the mold tends to wear out, deform, or lose precision due to deposits. Aside from a range of molds for products of different shapes and sizes, an inventory of spare molds needs to be maintained. This *repeated-use-limited-life* feature is also found in semi-conductor industry and printing industry in the form of masks and ink cartridges respectively. Additional examples are seen in general manufacturing environments; where expensive cutting tools, bushings, filtering equipment, spare parts etc. need frequent replacement due to wear or clogging. This category of resources is collectively referred to as *perishable resources*. In Chapter 3, the management of perishable resources is considered together with production planning and resource allocation or production

scheduling decisions. It is shown that these decisions are inter-dependent, and therefore, call for a combined optimization problem.

This new class of problems is solved using the widely accepted framework of hierarchical planning and scheduling using mathematical programming models. The results of this approach are compared (on several avenues) with the solution of the resource management problem when solved independent of production planning. A rolling horizon methodology is employed when the parameters like product demand and resource life have associated uncertainty.

However, when the system is plagued by high level of uncertainty, the performance of the rolling horizon approach is often unsatisfactory. To resolve this issue, the planning level problem is reformulated as a Markov decision process (MDP) in Chapter 4 and solved using an approximate dynamic programming (ADP) algorithm. The problem, in this form, resembles a popular class of problems called *dynamic resource allocation* problems (Powell 2005). However, the additional decisions related to production planning and resource procurement, introduce challenges in terms of problem formulation and determination of state transition function. Compared with the rolling horizon method, the ADP takes a more comprehensive view of the uncertainty, thereby, giving improved performances in all the stochastic models that are considered.

The focus is then shifted to the more popular category of manufacturing equipment, i.e., relatively bigger machines that are very costly to replace. The problem of devising an optimal preventive maintenance strategy for a single machine deteriorating randomly has been extensively studied (Monahan 1982). It is assumed that the machine condition degrades progressively with use in a non-self-announcing manner. This implies that the machine condition is not observed directly, but the degradation is reflected in increased production of defective items or reduced product quality. Inspection of the processed job, therefore, helps monitor the machine condition. But in the presence of high inspection costs, job inspection also becomes a part of the decision-making. This optimization problem has been successfully solved as a partially observable Markov decision process (POMDP), owing to limited observability of the machine state.

When the aforementioned machine/resource becomes a part of the manufacturing supply chain, it is bound to interact with other resources and possibly operate on multiple types of jobs. These possibilities are addressed in Chapter 5, by considering two process flow topologies: (i) re-entrant flow and (ii) a combination of re-entrant and serial flow topology referred to as hybrid flow. Due to costly inspection, not every processed job is tested. Consequently, the untested, potentially defective items move on to the next series of operations, thereby getting accumulated in the system, until removed in final product testing. The job inspection decision must now be taken to minimize this possibility in an economically favorable manner. This quality-control aspect is the single-most interesting addition to the optimization problem, together with the fact that, the resource management decisions for multiple resources are inter-dependent. The combined optimization problems for the two process flow topologies are solved as a (considerably larger) POMDP. Recent developments in the area of POMDP solution methods contribute greatly to the successful solution of the above problems. It is shown that the rigorous method using POMDP formulation has a large potential for improvement over heuristic rules.

Aside from machine maintenance, POMDPs have been successfully applied to a variety of stochastic problems with partial information. Their applications range from sensor allocation, robotics, network troubleshooting, moving target search etc (Cassandra, 1998). For this reason, POMDPs have received significant attention in the recent years (Spaan 2005; Thrun 2003; Simmons 2004). Since the exact solution methods are limited to very small sized problems, most research efforts have been spent on approximate solution methods. Approximate solution methods like point based methods, seek to perform Bellman updates on a subset of the state space (called belief space in this case). When the actions are continuous or have large number of dimensions (resulting in very large action spaces due to combinatorial reasons), the value updates are performed on a sampled set of actions. POMDPs with very large action spaces may also be solved using policy graph or policy iteration methods, but such methods are prone to local optima. In Chapter 6, an algorithm for the solution of POMDPs with high dimensional or continuous action spaces is developed. The Bellman updates are performed using a mixed integer linear program (MILP). MILP based Bellman updates can handle the continuous

or high dimensional actions while preserving the solution quality as opposed to the action sampling methods. The MILP based methods also provide better scalability with problem size. The concepts are illustrated by using a hypothetical POMDP with continuous actions followed by a network flow example. The latter corresponds to the case of discrete but high dimensional action space. The network flow problem also serves to represent yet another category of resource management problems, where the nodes of the network are prone to random contamination. This is a possibility in water, food and computer networks. Electrical networks, on the other hand, are prone to random outages. The network flow problem is solved using the existing enumeration based methods and the new (MILP based) solution algorithm. A comparison of solution times and solution quality are presented for both the examples.

In an alternative formulation, the MILP based Bellman updates are developed for POMDP around *post-decision* belief states. Formulation around post-decision belief states removes the dependence of MILP solution time on the size of the observation space. This enables the algorithm to be applied to a wider spectrum of POMDPs.

Chapters 1 and 2 are aimed at familiarizing the reader with the basics of planning and scheduling, resource degradation, inspection for diagnosis and quality control and existing models and methods. By formulation of problems and development of solution algorithms in Chapters 3 through 6, this thesis seeks to contribute an enhanced understanding, novel problems and efficient solution methods to the existing body of knowledge on (the general area of) resource management and related decision-making in various process industries.

CHAPTER 1

INTRODUCTION

1.1 Hierarchical decision-making in manufacturing

In a typical manufacturing environment, a detailed planning process is adopted in order to ensure the best utilization of resources and maximize a firm's profitability. This is done in a hierarchical fashion due to differences in time-scales and the impact of decisions constituting the planning process. (Anthony 1965) and later (Miller 2002) provide details on the concept behind Hierarchical Production Planning (HPP). The decisions are broadly classified into three categories.

Strategic planning:

At the strategic manufacturing planning level, the firm must address issues that bear a long term impact. Such issues comprise of the total planned production capacity levels for the next two, three, or more years; the number of facilities it plans to operate; their locations; acquisition of manufacturing and storage capacities and procurement of resources etc. Decisions made at the strategic production planning level place constraints on the next level of decision making, i.e., tactical planning level.

Tactical planning:

At this level, the chief goal of the decision-maker is to obtain and use the available resources effectively and efficiently. Typical planning activities include the allocation of capacity to various products, planning workforce levels, logistics of sourcing and distribution, preventive maintenance scheduling, and total quality management. The use of existing infrastructure is maximized while staying within the constraints of the firm's manufacturing and distribution infrastructure (as determined by previous strategic decisions). As with the previous level of decision-making, the planning decisions carried out at the tactical level impose constraints upon operational planning and scheduling decisions as discussed further. Typical planning horizons are 12 -18 months.

Operational planning:

The routine decisions related to shop floor control fall under this category. At this level, it is ensured that individual processes run efficiently and effectively. Master production scheduling, labor scheduling, process improvement, inspection and repair, truck load quantities, short term carrier selection etc. are a few examples of operational planning decisions.

At the lowest level of decision-making, i.e., the operational planning and scheduling level, most details about the individual processes are included. This leads to a high degree of complexity although the decisions have a very short term impact. As we move up the hierarchy, the system variables are aggregated. However, the risk associated with the decisions and their impact rises as we go from operational to tactical level and from tactical to strategic level. As noted in (Miller 2002) "A true HPP system is a closed loop system which employs a "top down" planning approach complemented by "bottom up" feedback loops. Given the emphasis of HPP systems on evaluating capacity levels and imposing and/or communicating capacity constraints form higher levels down to lower levels, it is imperative that strong feedback loops exist". This is because at the higher levels, possible infeasibilities are ignored or obscured due to aggregation. If the information about these infeasibilities is not communicated back to the higher levels, the firm may always function sub-optimally and may pay dearly. Together with strong feedback loops, a judicious scheme for aggregation of information is needed to construct the planning problems at various levels. This aspect often blurs the line between different levels of decision-making and results in an interesting inter-play between decisions at various levels. Particularly, in systems where there is a notable deterioration associated with manufacturing equipment, the classification of decisions into various levels of planning may present substantial challenges. Deterioration in manufacturing equipment is discussed further.

1.2 Resource degradation and related decision-making

Although resource is a general term, it is used in this work to specifically refer to a machine, equipment or tool group that facilitates production. In general, all manufacturing equipment is prone to degradation with time. The features listed below determine their impact on decisions related to manufacturing:

Type of degradation – The wear and tear associated with the usage of resource affects its performance. The degradation is generally reflected in falling product yields in flow type equipment, increased fraction of defective outcomes in discrete manufacturing (termed as process drift), larger number of non-conforming batches in batch processing etc. A complete failure or shutdown of the resource may also occur, leading to a halt in production. Yet another form of degradation is contamination of the resource to render it useless or even harmful for use. This is seen in network flow problems where the nodes of the network that facilitate flow of materials or information, have finite probabilities of contamination. Although, the nodes loosely fit the definition of a resource, it represents an important class of degradation management problems. In general, the degradation is caused by numerous factors including usage, age, type of operation, environmental conditions etc.

Corrective or preventive action – In view of the above mentioned deterioration, a preventive maintenance action needs to be taken to ensure equipment health. The frequency of the preventive action is largely dependent on the time scales associated with the degradation and trade-offs between the cost of maintenance and that of faulty products. Corrective action in the form of inspection and repair is required on the faulty outcomes. When the equipment breaks completely, it needs to be replaced or repaired. The downtime may encourage keeping spare equipment/resources. The choice of preventive and corrective actions is governed by industry, manufacturing process and costs involved with maintenance, repair and replacement.

Time scales – The period of time for which the equipment goes without showing signs of degradation is important to devise preventive and corrective actions. The time-scales associated with degradation are typically measured as expected time to failure or time

until the process yield falls to *x*% of the best possible yield. If the time-scales are very large, then the resource management decisions may be made independent of production decisions. However, if the time-scales are comparable with that of production, then resource failure or unavailability affects the production scheduling directly. In this case, the downtime associated with resource repair or maintenance must be accounted for in the production schedule.

Degradation dynamics – The dynamics associated with the degradation play an important role in resource maintenance decisions. The resource may degrade in a deterministic or stochastic manner. The variance associated with uncertain resource lifespan may be large. This warrants a more conservative preventive maintenance plan. Steep increases in defect fractions call for frequent product testing and resources prone to sudden and untimely failures require the presence of spares.

Upstream and downstream processes – The resource management decisions may not be taken independent of the rest of manufacturing supply chain. In general, the resources at the beginning of the production sequence need to be monitored more closely. This would avoid losses in terms of downtime during a failure and (or) propagation of faulty jobs to downstream processes.

Possibility of detection – Direct inspection of the equipment may be performed as part of routine check-up. When this is not possible or sufficient, inferential measurements on product attributes are taken to monitor the performance of the equipment. In situations where the quality is not reflected in process or control variables, installation of job-specific inspection stations or sensor networks is required for product testing. When equipment inspection or job inspection is costly, an inspection strategy becomes a part of the decision-making as well.

The knowledge of above factors is central to devising an efficient resource management plan and to establish whether resource management needs to be done in unison with other manufacturing related decisions like production planning and quality testing. As noted in above discussion, the resource failure rates may have associated uncertainty. This, together with other sources of uncertainty, poses challenges in

modeling and solution. The other sources of uncertainty and a brief discussion on solution methods are covered in the following section.

1.3 Uncertainty and observability

At different levels of the decision-making hierarchy, uncertainties present themselves in various forms. The presence of uncertainty results in several possible outcomes and future system states, often having far-reaching effects. Depending on the level and form of the uncertainty, it becomes imperative to factor it into the decision-making. This is because extreme realizations of uncertainty may result in great loss of performance, operational infeasibility or both. Uncertainty in the realm of planning problems may be classified as below:

Parametric uncertainty – When the problem parameters have randomness associated with them, the uncertainty is external to the system. The exogenous information about the parameters becomes available after the relevant decisions have been made. For example, the product demand for a firm's products is often uncertain and a buffer stock or safety stock is maintained to counter the effect of uncertainty. If the available stock falls short of extremely high realizations of demand, the stock-outs lead to loss of potential revenue and hamper the firm's reputation. Other examples of parametric uncertainty include market price for products, raw materials, utilities etc., raw material quality, conversion rates,

Decision Uncertainty – Often the outcome of a decision cannot be known with complete certainty because of the uncertainties associated with the process. For example, a decision is made to produce x units at a particular machine in a given time frame. However, due to random failure, only x-y, y>0 units could be made. In a different scenario, x units are decided to be manufactured by assuming an expected process yield of beta. This implies that a fraction of produced units will be found non-conforming by appropriate quality standards. However, a higher realization of beta results in lower number of conforming units. Robot movement, production throughput, corrective or repair actions, order quantities, sensor failure are a few more examples of decision uncertainty.

State uncertainty - When the elements of the state of the system are not known with complete certainty, a probability distribution is maintained over the state. In control applications, state uncertainty is generally attributed to measurement noise and an estimate of the state is maintained. In planning frameworks, the presence of this type of uncertainty is referred to as partial observability. For example, in a retail store, the inventory of products in the store is seldom known with certainty. Partial observability of the state is also a concern in preventive maintenance planning where the deterioration level of equipment is not completely observed. Errors associated with measurement sensors also leads to randomness in state estimates.

There is a fine line between the first two types of uncertainties in that they affect the future state in similar manners and the realization of uncertainty becomes available after an action is taken, i.e., the parameter values are realized or the effect of action becomes known. Therefore, both the parametric and decision uncertainties can be accounted for by similar modeling and optimization methods. However, in the case of state uncertainty, the realization of system's true state may never become available. Therefore, using the term uncertainty is a misnomer and it will be referred to as partial observability of state for future analysis. Specialized algorithms have been developed to solve optimization problems with partial state information.

Several techniques exist for solving general stochastic optimization problems:

- Stochastic dynamic programming (Bellman 1956; Puterman 1994).
- Mathematical programming methods like stochastic programming (Andrzej and Ruszczyński, 2003) and robust optimization.
- Simulation methods like simulated annealing (Kirkpatrick and Vecchi, 1983), genetic algorithms (Schmitt 2001) or Monte Carlo (Fishman 1996) simulation.

The applicability is largely governed by problem size, problem type (discrete or continuous states and actions) and search complexity. Most of the above methods optimize the expected value of the objective. Robust optimization considers the worst

case uncertainty by using a min-max approach while certain formulations like value at risk (*var*) or conditional value at risk (*covar*) also account for variance in the objective.

1.4 Outlook

As described in section 1.2, depending on the extent and type of resource degradation, other manufacturing decisions need to be taken in conjunction with the resource management decisions. For this purpose, three broad categories of resources are considered. The first category consists of small pieces of equipment that are present in large quantities on the production floor, and may move from one operation to the other. In the course of manufacturing, these resources are amenable to breaking (perishable resources), getting consumed (expendable resources) or exiting the system in some other way. Examples of such resources includes small fixtures of machines like masks, lenses in precision equipment manufacturing, substrates in catalysis, molds in building material industry, ink cartridge in printing industry etc. Human resources fall in this category when the employee turnover rates are high. This is true of industries like construction, military, call centers and consulting. The unique feature that differentiates the above from raw materials is that the resources can be employed multiple times before they exit the system. Since resources belonging to this category are generally required in relatively large quantities and the demand for them may not be well understood, an inventory is usually maintained. The demand for resources is determined by the production requirement and therefore, the inventory control at the product and resource levels becomes coupled. This problem can be viewed as a nested inventory control problem and is addressed in Chapter 3. In Chapter 3, mathematical programming models are developed for decision-making at planning and scheduling levels in a hierarchical fashion. Parametric uncertainty is handled by solving the planning and scheduling problem in a moving horizon fashion.

Depending on the its form, a myopic view of the uncertainty taken by the rolling horizon solution approach may not be very effective. For this reason, an approximate dynamic programming (ADP) algorithm is implemented to solve the perishable resource

inventory control in conjunction with production decisions. The solution from ADP is compared with that of the rolling horizon method.

The second category comprises the large machines which deteriorate gradually so that preventive maintenance needs to be performed on the machine/ resource from time to time. However, the degradation is prone to uncertainty and is seldom observed directly. Inferential measurements in the form of the quality of processed job are often taken to access the state of the machine. However, when product inspection is costly, it may not be economically favorable to test all the processed jobs and job inspection becomes a part of the decision-making. In a manufacturing system with multiple operations, the untested jobs move downstream for further processing. In the event that an untested job does not satisfy the quality requirements, this would result in propagation of defects, thereby raising quality control issues. This feature of defect propagation is addressed in Chapter 5 by means of two process flow topologies: (i) a re-entrant flow system and (ii) a hybrid flow system, in a discrete/batch manufacturing system. Due to lack of full information about the system at all times, e.g., the machine state, the problem is formulated and solved as a partially observed Markov decision process (POMDP). A comparison with alternative solution methods and an analysis of the solution properties is presented.

Over the last decade, considerable research efforts have been spent in the area of developing efficient algorithms to solve POMDPs. Since exact solution methods are limited to very small problem sizes, the focus is mainly on approximate solution methods. However, literature is relatively sparse on POMDPs with very large or continuous state, action and observation spaces. To this end, a mixed integer linear programming (MILP) based solution algorithm is developed in Chapter 6. In an alternative formulation, the POMDP solution is structured around the *post-decision* state variable to limit the effects of a large observation space. The methodology is implemented on a network flow problem, which comprises the third category of resources prone to degradation.

A network comprises of nodes and edges that facilitate the flow of material like water, food, electricity etc. or information like computer network or supply chain network etc. Occasionally, the nodes of the network get contaminated and the

contamination is amenable to spread to the downstream node. It is therefore imperative to track down and repair the contaminated/corrupted node and divert any flows so as not to pass through it. This problem is also formulated and solved as a POMDP using the MILP based algorithm. The solution quality and convergence times are compared with those of the traditional method.

CHAPTER 2

OVERVIEW OF DEGRADATION MODELING AND DECISION-MAKING UNDER UNCERTAINTY

2.1 Overview of degradation modeling, inspection strategies and solution frameworks

Degradation of manufacturing equipment/resources can manifest itself in many different ways. For example, with age or deterioration, the equipment might increasingly produce lower quality products, may cease to work, may corrode downstream equipment etc. A particular type of degradation of interest is when the gradual degradation of production equipment is reflected in increased production of off-specification products. In this case, the detection of quality attributes of the products requires job inspection by means of measurement sensors. Literature is replete with studies on inspection allocation in manufacturing environments. Along with degradation management, much of the work in the past has gone into detecting the degradation by means of correct inspection strategy. Serial manufacturing systems have been most popular means of illustration of the concepts. A serial manufacturing line has sequential operations and the product flow is linear. The sensor allocation is performed with one of the two major objectives; namely, inspection oriented quality assurance policies that are aimed at product improvement by rework/ repair and the diagnosis- oriented sensor distribution strategies which are focused on diagnosing the deteriorating process/equipment. The two broad problem classes are discussed in further details.

2.1.1 Inspection-oriented quality assurance strategies

The inspection-oriented quality assurance strategies may be viewed as a corrective measure to ensure that the faulty products do not reach the end customers. These account for the trade-offs between the costs associated with inspection stations and the returns obtained by the improved quality of processed jobs. Some of the early works in this area are reported in (Raz 1986). Most of the authors that were cited, consider serial

manufacturing lines for inspection allocation due to ease of analysis. The literature on inspection allocation problems, since then, is comprehensively reviewed in (Mandroli et al, 2006). A typical serial production line has single or multiple types of jobs, all with a predetermined sequence of operations. The allocation problem entails determining the optimal locations of inspection stations when there is limited inspection capacity. Inspection capability may also be limited by cost being substantial if it was to occur after every operation. The defect level found upon inspection can be a binary or continuous measure, and different decisions like rework repair, replace or scrap are taken depending on the production stage and extent of defect. The intensity of inspection, e.g. selective inspection and repeated inspection is also a decision to be considered. The inspection may be prone to imperfections and may have type I or type II errors as explained below:

(i) the wrong rejection of a conforming unit (type I error)
(ii) the erroneous acceptance of a non-conforming unit (type II error)

The jobs may have multiple defect types as caused by different operations. The inspection stations may or may not be able to detect all defect types caused in the past. In the absence of the ability to detect all types of defects, the inspection is termed *specialized*. A sub-set of above mentioned issues have been considered in literature, which is briefly reported here.

Earliest work in this area was conducted by (Lindsay and Bishop, 1964) who formulated a dynamic programming problem for basic issues related with inspection allocation. They also proved that an extreme point solution (0% or 100 %) is optimal, if it is assumed that all rejected items were scrapped, unless we have a constraint on final product quality level. The extreme point solution boils down to inspecting all jobs whenever an inspection station exists after an operation. A screening inspection program for a multistage process can be established by considering three related decisions. These include the location of inspection station, the level of inspection at each inspection point and types of inspection at any stage for specialized inspection stations. This concept of specialized inspection problem was further extended by (Rebello et al, 1995) who presented some exact and heuristic solution methods. (Lee and Unnikrishnan, 1998)

extended the above mentioned decision making to multiple part types which have different sequences of operation. They also relaxed the perfect inspection assumptions made by the above mentioned works and included inspection errors. Shiau (2003) assigned dynamic tolerances for inspection, by assuming a random defect generation model, and proposed heuristics for solution of the resulting large size problem. Kakade et al, (2004) proposed a simulated annealing approach to capture economic tradeoffs between product yield and inspection accuracy. Gurnani et al, (1996) integrated the inspection allocation problem with that of capacity planning and inventory levels. Another aspect of inspection allocation is dedicated to equipment diagnosis and process improvement. This aspect is discussed in the following section.

2.1.2 Diagnosis-oriented sensor distribution strategies – Process improvement

To facilitate process improvement or maintenance scheduling, sensor distribution strategies must have a deeper insight into the measurements of faults and defects. The process variables and quality variables that provide the information about machine health are the target of this study. The decisions involved in this type of study are

(i) The workstations where to place the sensors
(ii) The physical variable to be measured at each station
(iii) Equipment maintenance decisions – service or replace the machine

Mandroli et al, (2006) reviewed existing literature on quality-fault modeling and effectiveness of sensor systems. For optimal allocation in this case, a measure of effectiveness is maximized, the overall cost is minimized or yield is maximized with constraints on sensor allocation. Optimization approaches like Powell's direct search (Wang and Nagarkar, 1999), sequential quadratic programming or gradient-based search (Khan et al, 1999), exchange algorithms (Liu et al, 2005), have been used in the existing literature. Nurani el al, (1994) developed an optimal sampling plan specifically in semi-conductor fab using a statistical process control approach.

The sensor allocation is followed by machine/ process improvement. To this end, Rabinowitz and Emmons, (1997) proposed a setting where defects provide full

information about health of the machine and machine is repaired whenever it is detected to be malfunctioning. Expanding the work, Emmons and Rabinowitz, (2003), the authors proposed heuristics methods for inspection task scheduling in general deteriorating systems. Bowling et al, (2004) prescribed optimum process targets using a Markovian approach, while Yacout and Gautreau, (2000) included partial observability in their analysis. Yao et al, (2004); Marcus et al, (2004) considered the impact of production constraints on preventive maintenance (PM) scheduling in manufacturing systems. They developed a hierarchical method to obtain a time window within which PM must be scheduled. Cassady and Kutanoglu, (2005) integrated PM scheduling and production scheduling in an MILP framework.

2.1.3 Existing sensor technology and defect and variance propagation models

To be able to mathematically analyze the system, we need to model the defect generation by various operating stations/ machines. The model should be able to capture the propagation of defect in multi-stage setting and sources of variations at a particular station. Also, for process improvement and sensor distribution, appropriate process variables must be included in the model. Cochran and Erol, (2001) proposed analytical methods to model process flows, and performance measures like outgoing quality level and throughput rate. Zantek et al, (2002) established modeling techniques for correlated stages and estimated the parameter values by least squares method. Chan and Spedding, (2001) captured the propagation of defectives by design of experiments (DoE), response surface plot and a neural network model. State-space models have been frequently used to characterize the variance (Ding et al, 2000; Huang and Shi, 2004). Huang and Shi, (2004) used a state-space model to capture the propagation of variance in serial-parallel multistage systems. To detect the source of variance in processes, Lee and Apley, (2004) used linear structure model to generically represent the variation patterns. Batson (2004) described a simple probabilistic model that describes the serial effects on the processed parts. The cost of less-than perfect quality was approximated by a quadratic loss function called *Taguchi loss function*.

To determine target applications for our methods, we reviewed the status of sensor technology in a discrete part manufacturing setting, for example, semi-conductor

manufacturing. Grochowski et al, (1997) summarized the current status and trends in integrated circuit testing. Kumar et al, (2006) reviewed the various yield modeling techniques specifically in semiconductor manufacturing. Xiong et al, (2002) successfully applied the modeling techniques for variation prediction in automotive assembling.

2.1.4 General integrated control and optimization framework

Given all these different aspects of decision making for inspection allocation and process improvement, there is a need to integrate them into a single decision making framework for optimal decisions. One example of such a framework is the fab-wide control framework developed for semiconductor manufacturing processes (Qin et al, 2006). The authors used a hierarchical framework for integrated control. In the fab-wide framework that they proposed, at the bottom of the hierarchy lie the run-to-run controllers which optimize the local processing steps. These run-to-run controllers are controlled by an 'island of control' which in turn is supervised by an overall fab-wide controller satisfying the economic goals. A similar integrated control framework was suggested by Tosukhowong (2006) for continuous flow process plants. Similar methodology was adopted by Vargas-Villamil et al, (2003), where use of Model Predictive Control (MPC) is made at the intermediate layer. Sethi and Zhang, (1994) discuss hierarchical decision making in stochastic manufacturing systems which brings together optimal control of parallel machine and dynamic flow shops. Very few attempts have been made at using frameworks that are different from hierarchical control frameworks. One such work (Heragu et al, 2002) proposes an intelligent agent based framework which is a hybrid of the hierarchical and heterarchical frameworks. They used the concept of holonic structures that accomplish individual as well as system wide objectives. Inman et al, (2003) considered the intersection of quality and production system design and suggested new research issues regarding trade-offs between productivity, flexibility and quality in manufacturing environment.

2.1.5 Techniques for solution

Since most of the above problems are multistage, Dynamic Programming (DP) has been widely used as a solution approach for these problems (Lindsay and Bishop,

1964; Gurnani et al, 1996). Bertsekas (1995); Sutton and Barto, (1998) contain comprehensive discussion on exact and approximate solution methods of the DP problem. Many of the above problems can be formulated as Markov Decision Processes (MDPs) in which transition between states follows Markov property. Quantitatively, this is a 'memory-less' property where system state and stage-wise reward only depend on the one step prior state and action. Occasionally, the system state information is missing/hidden causing it to be a Partially Observable Markov Decision Process (POMDP). Other than DP, mixed-integer programming (MIP) and non-linear programming (NLP) formulations have been successfully used in inspection allocation problems (Rebello et al, 1995). Several heuristic methods like simulated annealing, genetic algorithms, random search methods and simulation are used when problem size is big enough to inhibit usage of exact solution methods. Some of the properties that heuristics (*rules of thumb)* must capture are as summarized in Lee and Unnikrishnan, (1998):

- Inspect before costly operations so that these operations will not be performed on non-conforming items
- Inspect before items that cover up/obscure non-conformities
- Inspect before operations where faulty items may jam or break the machines

Since MDP and POMDP have proven to be successful frameworks for solving problems related to inspection, maintenance and production scheduling, the two frameworks are reviewed in greater detail.

2.2 Markov decision processes

Markov Decision Processes (MDPs) provide a framework for modeling real world processes which have a stage-wise structure. The stage can denote a time epoch or other quantities like location, processing step etc. At any stage, the system is recognized as being in a state (designated as s) which is a set of attributes that aid decision-making. The set of all possible states is called state space (designated as S). Starting in state $s \in S$, there

is a set of actions from which the decision-maker must choose. The set of all possible actions is called action space (A) and an element of the action space is denoted by a. When action a is taken in state s, and the system transitions to the next stage, it ends up in a unique next state $s' \in S$ in the absence of any uncertainty. However for stochastic problems, there is a set of possible next states for each state-action pair. The probability of transition to a particular next state, in this case, is governed by a state transition probability function T. In the process, reward $r(s,a,s')$ is received, which is determined by the reward function R. The dependence of r on s' is often suppressed by taking a weighted average over all possible states at the next stage. At each stage, actions are taken so that the sum of stage-wise rewards is maximized. In the presence of uncertainty, the expected sum of rewards is maximized. When infinite stages are present, i.e., extremely large time horizon, the future rewards are often discounted using a discount factor γ. When the number of stages is infinite, the problem is called an infinite horizon MDP as opposed to finite horizon MDP for finite number of stages. In most applications, a stage symbolizes a time epoch. Therefore, we use the term *time epoch* or *time step* synonymously with 'stage' for future reference.

$$V^{\pi}(s) = E_{\pi}[\sum_{t=0}^{\infty} \gamma^{t} r(s_t, \pi(s_t)) \mid s_0 = s] \tag{2.1}$$

More formally, a MDP corresponds to a tuple (S, A, T, R, γ) where S is a set of states, A is a set of actions, $T : S \times A \times S \rightarrow [0,1]$ is a set of transition probabilities that describe the dynamic behavior of the modeled environment, R: $S \times A \times S \rightarrow \mathbf{R}$ denotes a reward model that determines the stage-wise reward when action a is taken in state s leading to next state s' and $\gamma \rightarrow [0,1]$ is the discount factor used to discount future rewards. A γ value close to 0, places very little weight on future rewards, while γ close to 1 results in very little discounting.

The notational convention for MDPs is adopted from Hauskretch (2000) with small modification. For ease of illustration symbol $p(.)$ is used to denote probability of a quantity and $r(.)$ is used to denote reward (generally as a function of state and action). p_{ij} signifying transition from state $s=i$ to $s'=j$ is used to denote transition probabilities

associated with the Markov chain. T_a is used to denote the probability transition matrix corresponding to action a. Symbols s, s' and a are used to denote current state, next state and action and belong to sets S,S and A respectively.

One of the fundamental properties of the MDPs is that the transition and reward functions associated with the stage-wise transition of state are independent of the past states and actions. Referred to as Markov property, this memory-less feature enables the decomposition of the overall optimization problem into separate stage-wise problems. This is accomplished by using a recursive relationship between the value of being in a state at any stage.

The goal is to maximize the (often discounted) sum of rewards over a time horizon which can be either finite or infinite (2.1), where t denotes the time epoch, s_t is the state at time t and π: $S \rightarrow A$, is the policy that dictates the choice of action at time t. This is achieved by solving the Bellman equation (Bellman 1956) for finite or infinite horizon problems (2.2). It is well-known (Puterman 1994) that for infinite horizon problems, a stationary optimal policy of the form in (2.3) exists, where $V^*(s)$ is the average discounted infinite horizon reward obtained when the optimal policy is followed starting from s until infinity (Puterman 1994). This implies that the state to action mapping, in the form of optimal policy is independent of the time epoch. The existence of stationary optimal policy is conditioned on the properties of model elements. One of the sufficient conditions is that there be a finite action space A_s corresponding to each state $s \in S$, maximum attainable stage-wise reward is finite and discount factor $\gamma \in [0,1)$. For all applications in this work, the set of conditions noted here are satisfied. The alternative sets of sufficient conditions for existence of a stationary optimal policy for discounted infinite horizon MDPs can be found in Puterman (1994). In (2.3), $a^*(s)$ is the optimal action to be taken when the system is in state s, independent of time t. $V^*(s)$ is called the optimal value function and is obtained as the solution to Bellman equations (2.2) for all s.

$$V^*(s) = \max_{a \in A} \{ r(s,a) + \gamma \sum_{s' \in S} p(s' \mid s,a) V^*(s') \} \qquad \forall s \tag{2.2}$$

$$a^*(s) = \arg\max_{a \in A} \{ r(s,a) + \gamma \sum_{s' \in S} p(s' \mid s,a) V^*(s') \} \qquad \forall s \tag{2.3}$$

It must be noted that the set of Bellman equations (2.2) (also called optimality equations), are difficult to solve analytically because of the presence of max operator. For the solution of MDPs in this work, one of the popular solution methods called value iteration (Puterman 1994) is chosen. The algorithm is shown in Figure 2.1. Starting with an arbitrary value function $V_0(s)$ for each state $s \in S$, the value function is iteratively improved using (2.4) until ε-convergence is reached. Subscript n denotes the iteration counter. The operator for one iteration can be denoted as H such that $V_{n+1} = HV_n$. The sequence of estimates of value function $V(s)$ for $s \in S$ converges to fixed point solution. This is a consequence of Banach's theorem for contraction mappings (Puterman 1994). Since H is a proven contraction map, the convergence properties hold.

Step 0. Set $V_0=0$ for all $s \in S$
fix a tolerance parameter $\varepsilon>0$
set $n=1$

Step 1. For each $s \in S$ compute:

$$V_{n+1}(s) = \max_{a \in A} \{ r(s,a) + \gamma \sum_{s' \in S} p(s' \mid s,a) \gamma V_n(s') \} \qquad \forall s \tag{2.4}$$

$$a_{n+1}(s) = \max_{a \in A} \{ r(s,a) + \gamma \sum_{s' \in S} p(s' \mid s,a) \gamma V_n(s') \} \qquad \forall s \tag{2.5}$$

Step 2. If $| V_{n+1} - V_n |_\infty \le \varepsilon$
set $a^\varepsilon = a_{n+1}$, $V^\varepsilon = V_{n+1}$ and Stop
else, set $n=n+1$ go to Step 1

Figure 2.1: Value iteration algorithm for solution of MDP

Due to ease of implementation, value iteration is perhaps the most widely used algorithm in dynamic programming. Certain other methods like policy iteration (Bertsekas 1995), a hybrid between value iteration and policy iteration (Powell 2007) and linear programming method for dynamic programs (De Farias and Roy 2003) are also used depending on the problem structure. The complexity of the algorithm shown in Figure 2.1 grows as a function of $o(|S|^2|A|)$. This is attributed to the three curses of dimensionality noted below:

1. Equation (2.4) needs to be solved for all s belong to S, so the solution time is directly proportional to $|S|$
2. The complexity of max operation depends on the size of the action space $|A|$
3. The calculation of expectation within the max operator depends on the number of possible next states, i.e., $|S|$.

In the presence of large state and (or) action spaces, the value iteration algorithm cannot be implemented in its exact form. Several approximation methods have been developed to circumvent this difficulty. Some of them being, approximate dynamic programming methods using value function approximations (Powell 2007), Q-learning, temporal difference learning (Barto et al, 1995; Sutton and Barto, 1998), linear programming methods using basis functions (De Farias and Roy, 2003) and dynamic programming methods using post decisions state (Powell 2007). Details of particular approximate solution methods used in this work are deferred until the specific illustrations/applications for better understanding. All the above methods assume that the system state is completely known or observed at all times. When this assumption does not hold, the equivalent framework is called a partially observed Markov decision process (POMDP) as discussed in the next section.

2.3 Partially observed Markov decision processes (POMDP)

POMDP is a discrete-time stochastic control process when the states of the environment are partially observed. Similar to the MDP, at any time, the system is in one of the states s in the state space S. By taking an action a, the system transitions to the next state $s' \in S$ according to known system dynamics $p(s'|s,a)$ and accrues a reward $r(s,a)$. The

next state s' is not completely observed but an observation o may be made which is probabilistically related to the state s' and action a by $p(o|s',a)$.

More formally, it corresponds to a tuple (S, A, Θ, T, O, R) where S,A,T,R and γ represent the same entities as described in section 2.2 on MDPs; Θ is a set of observations, and $O : S\times A\times \Theta\rightarrow[0,1]$ is a set of observation probabilities that describe the relationship among observations, states and actions. The notational convention for POMDPs is also adopted from (Hauskretch 2000) for consistency. Symbol $o\in\Theta$ is used to denote an element of the observation space Θ.

When the system state s is not perfectly observed, a history of all actions and observations (since t=0) need to be maintained. Due to the Markov property, this information is contained in the probability distribution over all states at any time. The probability distribution is referred to as belief state $b(s)$ for $s\in S$. The belief states are continuous since they contain the probability values, which are continuous numbers between 0 and 1. Partial observability, thus converts the original problem into a fully observable MDP (FOMDP) with continuous states. Since all the elements of a belief state must add up to 1, the state dimension of FOMDP is one less than the size of the original state space.

Similar to MDP, an infinite horizon POMDP has an optimal stationary policy $\pi^* : \Delta\rightarrow A$ which maps the *belief* states to optimal actions (Smallwood and Sondik, 1973). $\Delta : R^{|S|-1}\in(0,1)$ is the belief simplex containing all possible belief states. A policy π can be characterized by a value function V^π which is defined as the expected future discounted reward. $V^\pi(b)$ is accrued when the system is initially in state b and policy π is followed (2.6), where $0\le\gamma<1$ is the discount rate that discounts the future rewards.

$$V^{\pi}(b) = E_{\pi}[\sum_{t=0}^{\infty}\gamma^{t} r(b_t,\pi(b_t)) \mid b_0 = b] \tag{2.6}$$

The value function corresponding to the optimal policy maximizes $V(b)$ and satisfies the Bellman equation (2.7) for all b.

$$V^{*}(b) = \max_{a\in A}[\sum_{s\in S} r(s,a)b(s) + \gamma\sum_{o\in O} p(o\mid a,b)V^{*}(b_a^o)] \tag{2.7}$$

where, b_a^o is the belief state obtained when action *a* is taken in state *b* and observation *o* is made. The expression for b_a^o is shown in (2.8).

$$b_a^o(s') = \frac{p(o \mid s', a) \sum_{s \in S} p(s' \mid s, a) b(s)}{p(o \mid b, a)} \tag{2.8}$$

Similar to the value iteration for MDPs, the value update step for a belief point *b* is shown in (2.9).

$$V_{n+1}(b) = \max_{a \in A} [\sum_{s \in S} r(s, a) b(s) + \gamma \sum_{o \in O} p(o \mid a, b) V_n(b_a^o)] \tag{2.9}$$

However, exact value iteration may not be performed due to the presence of continuous states and consequently, infinite number of belief states. To alleviate this problem, researchers have looked into ways to exploit the fact that the optimal value function corresponding to POMDPs has a parametric form. For finite horizon problems, the value function is piece-wise linear and convex (PWLC) (Smallwood and Sondik, 1973) and for discounted infinite horizon POMDPs, it can be approximated well with a PWLC function (Sondik 1978).

The POMDP solution methods can be broadly classified into the following categories (Hauskretch 2000; Spaan and Vlassis, 2005):

1. Exact solution methods:

   Exact methods of solution of POMDPs were developed in 1970s and are still being improved. Notable among these are enumeration (of all possible linear functions) and pruning (Sondik 1978; Monahan 1982; Cassandra et al, 1997; Zhang and Lee, 1998; Zhang and Liu, 1997). Sondik's one and two-pass algorithms (Sondik 1978) and the Witness algorithm (Kaelbling et al, 1999; Littman et al, 1995; Cassandra 1998).

   However, the exact solution methods are limited to very small size problems. (Papadimitriou and Tsitsiklis,1987) demonstrate that solving a POMDP problem is an intrinsically hard task. Finding the optimal solution for the finite-horizon

problem is PSPACE-hard and finding the optimal solution for the discounted infinite horizon criterion is even harder. The corresponding decision problem has been shown to be undecidable (Madani et al, 1999), and thus the optimal solution may not be computable.

2. Heuristic methods for value function approximation:
   Several methods have been proposed for approximation of the value function corresponding to the POMDP, e.g., approximation using the MDP value function (Astrom 1965; Lovejoy 1993), approximation using MDP Q-function (Littman et al, 1995) the fast informed bound Method (Hauskretch 2000), grid based approximations using interpolation by convex rules or curve fitting.

3. Finite state controllers or policy graph methods (Kaelbling et al., 1999; Littman 1996; Cassandra 1998)

4. Point based methods.

Over the years, many methods have been developed that make use of the PWLC structure of the value function to solve the POMDPs. Since, the exact solution methods are limited to problems of very small sizes, approximate point based solution methods like PERSEUS (Spaan and Vlassis, 2005), HSVI (Smith and Simmons, 2004), BPVI (Pineau et al, 2003) etc. have been studied recently, which expand the scope of POMDPs to problems of much larger sizes. PERSEUS is one of these methods, which uses the concept of asynchronous dynamic programming and randomly updates only a subset of belief states in one value iteration step. In this work, value updates in spirit similar to PERSEUS are used. The algorithm is described in further detail below.

2.3.1 PERSEUS – an approximate solution method (Spaan and Vlassis, 2005)

Given the PWLC structure of the value function, the value function at the n^{th} iteration (V_n) is parameterized by a finite set of gradient vectors α_n^i, i= 1, 2, .., $|V_n|$ (2.10). The gradient vector that maximizes the value at a belief state b (also referred to as a

belief point or just point) in the infinite belief space is represented by $\alpha_n^{(b)}$ in (2.11). Superscript *i* indicates the i^{th} gradient vector in the set and superscript (*b*) indicates the vector that maximizes $V_n(b)$ for a particular *b*. During an exact value iteration step then, the value ($V_{n+1}(b)$) and the gradient ($\alpha_{n+1}^{(b)}$) corresponding to any point can be updated using the Bellman backup operator as shown in (2.12) through (2.14).

$$V_n(b) = \max_{\alpha_n^i} b.\alpha_n^i \tag{2.10}$$

$$\alpha_n^{(b)} = \arg\max_{\alpha_n^i} b.\alpha_n^i \tag{2.11}$$

$$backup(b) = \alpha_{n+1}^{(b)} = \arg\max_{\{g_a^b\}_{a\in A}} b \cdot g_a^b \tag{2.12}$$

where,

$$g_a^b = \sum_{s\in S} r(s,a)b(s) + \gamma \sum_{o\in\Theta} \arg\max_{\{g_{a,o}^i\}_i} b \cdot g_{a,o}^i \tag{2.13}$$

$$g_{a,o}^i(s) = \sum_{s'\in S} p(o \mid s', a)\, p(s' \mid s, a)\alpha_n^i(s') \tag{2.14}$$

$$V_n(b) \le V_{n+1}(b) \le HV_n(b) \quad \forall b \in B \tag{2.15}$$

In PERSEUS, a subset *B* of belief points is obtained by taking random actions. This belief set is fixed and chosen as the new belief space for value function updates. Due to parameterization of the value function (2.10), an updated gradient vector for a belief point may improve the value of many other points in the belief set. This leads to the concept of approximate PERSEUS backups as shown in the algorithm below. Due to this approximate update, in each value backup stage, the value of all points in the belief set can be improved by updating the value and gradient of only a subset of points. The resulting value function estimate will follow the condition shown in (2.15) where HV_n is the estimate, if the entire belief space were updated.

Perseus backup stage: $V_{n+1} = H_{perseus} V_n$

1. Set $V_{n+1} = \emptyset$. Initialize $\tilde{B}$ to B
2. Sample a belief point b uniformly at random from B and compute α = backup(b)
3. If $b.\alpha \geq V_n(b)$ then add α to V_{n+1}, otherwise add $\alpha' = \arg\max_{\{\alpha_n^i\}_i} b.\alpha_n^i$ to V_{n+1}.
4. Compute $\tilde{B} = \{b \in B: V_{n+1}(b) < V_n(b)\}$.
5. If $\tilde{B} = \emptyset$ then stop, else go to 2.

PERSEUS is an elegant and fast method for solution of POMDPs with proven convergence properties. For convergence, it is required that the initial value function is always under-estimated everywhere. However, there are no performance guarantees with respect to the optimal value function. This is because the method considers a randomly selected belief set on which value iteration updates are carried out. This is done under the assumption that the parameterization using the gradient vectors would generalize well to the entire belief space. However, there is no indication of how good that generalization will be, even after the convergence criterion is met. Therefore, re-sampling techniques are used to ensure that the value function generalizes well to different parts of the belief space.

With the basic understanding of related literature on formulation and solution of manufacturing related problems in the face of resource degradation, a planning and scheduling problem with non-stationary resources is presented in the following two chapters.

CHAPTER 3

PLANNING AND SCHEDULING WITH PERISHABLE NON-STATIONARY RESOURCES

3.1. Introduction

Decision-making in manufacturing occurs at multiple levels ranging from high level planning decisions to low level shop-floor/ scheduling decisions. In the presence of different time scales, the decisions are taken in a hierarchical fashion to overcome the intractability of a combined large problem. Nevertheless, the controls/ decisions at different levels of decision-making affect one another.

This chapter presents a new class of planning and scheduling problems: *the perishable resource problem* where the upward flow of information (from the scheduling level to the planning level) is more significant as compared to traditional formulation. The term '*perishable resource*' stands for resources (machines or equipment) that can break frequently, and whose rate of breaking is a function of their use. When large numbers of each type of resource are present, an inventory of resources needs to be managed by suitable reorder and allocation policies. However, the traditional methods of inventory control may not be applied directly because the demand for new resources is governed by the resource allocation and production decisions.

Such systems are found, for example, in the building materials, semiconductor equipment and printing industries. The concepts developed may also be applied to a general resource management problem where resources exit the system at time scales comparable to that of production. A good example is workforce hiring, training and staffing decisions in an industry with high employee turnover rates. Construction, military, call center and consulting are a few examples of industries with high employee turnover. We have chosen the production of *culture stone/ stone veneer* for illustration purposes. The resources considered in this study have the following attributes:

Nomenclature

General

n_P –Total types of products (-)

C – Capacity of the production facility (no. runs/ day)

r - Number of time periods after which a demand scenario is repeated in a cyclical demand pattern

θ_T – terminal age of a new resource

R_i – Revenue from sale of product i ($/unit)

P_i – Production cost of product i ($/unit)

E_j – Equipment cost of resource of type i ($/unit)

H_i – Holding cost of product i ($/unit)

H_{ij} – Holding cost of resource of type i and remaining runs j ($/unit)

D_i^k - Actual demand for product i at the end of planning period k

D_{it} – Projected demand for product i at the end of planning period t. (-)

l – Resource replenishment lead time

Planning LP variables

H- Planning time horizon

x_{it} – Product Inventory (number of products of type i in the inventory at the end of planning period t)

$x1_{it}$– Production quantity (amount of product i produced during planning period t)

$x3_{it}$– Product sales (sale of product i at the end of planning period t)

y_{ijt}– Resource inventory (inventory of resources with remaining runs j, dedicated to product type i, at the end of planning period t)

$y1_{ijt}$– Resource reorder (number of new resources dedicated to type i with remaining runs j to be brought into the system at the beginning of planning period t)

$y2_{i,i1,jt}$– Resource transition (number of resources of type i with remaining runs j to be transitioned to type $i1$ during planning period t)

$y3_{ij,j1,t}$– Resource allocation (number of resources of type i with remaining runs j that are scheduled for j-$j1$ runs during planning period t)

x_i^0 - Initial product inventory at the beginning of the planning horizon (-)

y_{ij}^0- Initial resource inventory at the beginning of the planning horizon (-)

$y1_{ij,t}^0$- Pre-ordered quantities for resource of type i with remaining runs j scheduled to arrive at the beginning of t (-)

Scheduling LP variables

h – Scheduling time horizon

$\tilde{x}_{it}$ – Product Inventory (number of products of type i in the inventory at the end of scheduling period t)

$\tilde{x}1_{it}$ – Production quantity (amount of product i produced during scheduling period t)

$\tilde{y}_{ijt}$ – Resource inventory (inventory of resources wiht remaining runs j dedicated to product type i, at the end of scheduling period t)

$\tilde{y}2_{i,i1,jt}$ – Resource transition (number of resources of type i with remaining runs j to be transitioned to type $i1$ during scheduling period t)

$\tilde{y}3_{ijt}$ – Resource allocation (number of resources of type i with remaining runs j that are scheduled for a run during scheduling period t)

x_i^0 - Initial product inventory at the beginning of the scheduling horizon (-)

y_{ij}^0- Initial resource inventory at the beginning of the scheduling horizon (-)

$penalty1_i$ – Penalty for missing target inventory of product i prescribed by the planning level

$penalty2_i$ - Penalty for missing target inventory of runs from resource of type i prescribed by the planning level

$w1$ – weighting factor for $penalty1$

$w2$ – weighting factor for $penalty2$

- Quantity – there are multiple resources simultaneously being used in the system at a given time.
- Reusability – the resources can be used multiple times before they get consumed, break or exit the system.
- Flexibility – one resource is dedicated to one job for a given production period. However, the resources can be assigned to multiple types of jobs by incurring transition expense in terms of cost, time or both.
- Mortality – the time scales associated with the consumption/breaking of the resources are comparable with that of production.

Due to the above mentioned characteristics, a perishable resource combines features of a conventional piece of equipment (reusability, flexibility) and those of raw materials (quantity, mortality). However, unlike raw material planning, the consumption equation of the perishable resource as a function of production is complicated because of the reusability aspect. Also, the time scales of breaking being comparable with that of production, the inventory control problems at the product and resource levels need to be addressed together.

Consequently, in conjunction with the resource management decisions, production planning and scheduling decisions need to be made. In such a scenario, the typical objectives for the decision-maker, in the order of importance, are:

- Keep up with the uncertain demand for product (production planning)
- Have an optimal resource re-order policy in the face of uncertain resource life, to satisfy the production schedule (resource-reorder)
- Minimize the breakage rate of the resource by optimal allocation of resource (optimal resource allocation)

These objectives lead to a bi-layer problem, as shown in Figure 3.1, with resource reorder and production planning as the high level decisions in monthly buckets while the resource allocation as the low level decision in daily buckets. Inventory is maintained at two levels in the supply chain, namely the inventory of products and that of resources. At

the planning level, the target inventory of products and resources at the end of each planning cycle are prescribed, while at the scheduling level, a detailed production schedule is generated by allocating the resources to required jobs. Since the resources have finite lives, resource life distribution contains information about how many resources are going to (or likely to) break in the next planning cycle. This information is communicated to the planning level at the beginning of each planning cycle.

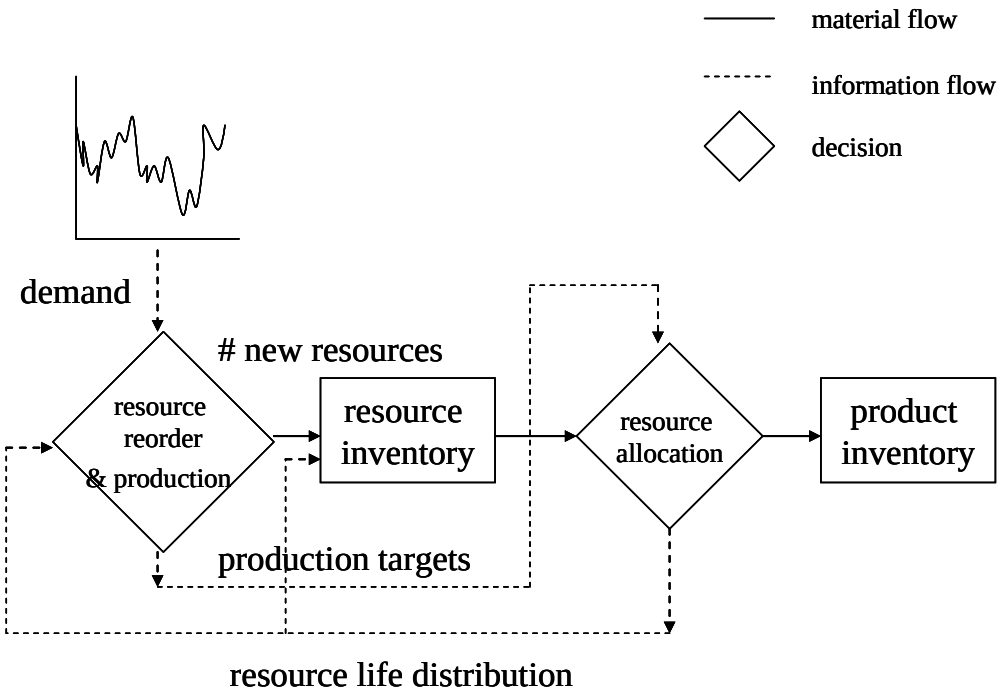


Figure 3.1: Flow diagram for the bi-layer inventory control problem

It must be noted that uncertainty in parameter values can be explicitly accounted for by using stochastic optimization methods like dynamic programming or stochastic programming. However, these methods soon become intractable with increased problem size. A discussion on stochastic treatment of the perishable resource problem is deferred until the next chapter. For formulation purposes, related literature on perishable inventory

problem (Haijema et al, 2004), flexible resource allocation (Harrison and Meighem, 1999) and general resource allocation problems (Powell 2007) was reviewed.

Perishable resource problem is distinguished from perishable inventory problem or a general resource allocation problem (Powell 2007) by at least two complicating features: (i) the demand for resources is determined by production decisions; therefore uncertainty is governed by decisions, (ii) there is a strong correlation between the inventories of products and resources, which need to be controlled simultaneously. For solution methods, the existing theory on inventory control (Hopp and Spearman, 1996) and hierarchical treatment of planning and scheduling problem (Miller 2002) was studied.

This chapter is organized as follows. For ease of understanding the formulation and notation, the *stone veneer* example is presented in the next section. However, the concepts and notation are fairly general for systems involving perishable resources. The solution methods for scheduling and planning levels are discussed in sections 3.3, 3.4 and 3.5. Results and discussion are presented in section 3.6.

3.2. Problem description through *stone veneer* supply chain

Stone veneer is an artificial stone used in building facades, gardens, etc. for decorative purposes. It is made by pouring a concrete mixture into molds and letting it cure. These molds facilitate the production of stones of various shapes and colors. The molds of a particular shape are painted in different colors to make a variety of n_P different color grades. A mold can only be used once a day, and the facility has limited capacity (C) in terms of number of molds processed per day. The processing of one mold is referred to as one 'run'. There are multiple molds in the facility at a given time and these molds are the resources that are prone to breaking. Therefore, an inventory of molds needs to be maintained, to satisfy the production needs. In the presence of product demand uncertainty, stones of various color grades are also kept in store. Finally, the transition of mold from one color grade to another may lead to further degradation and eventually cause the mold to break more quickly. The transitions of molds from one color grade to the other are uni-directional. For example, only transitions from molds of light

color grades to dark are allowed. Therefore, the more the fraction of dark molds in the system, the more inflexible it is in terms of color grade allocation of molds.

In summary, inventory is held at two points in the system (as in Figure 3.1) and both affect one another. When the molds of a certain color grade fall short of the demand, the decision-maker may either perform a color grade transition of existing molds at the expense of flexibility of allocation, or procure new molds at added cost and increased mold inventory. This trade-off is a unique aspect of this study, coupled with the need to keep up with product demand. The modeling details on resource age and product demand are presented in the rest of this section along with the objectives and decisions.

3.2.1 Modeling the resource age

The resource degrades due to general usage and due to transitions from one color grade to the other. The mold life is modeled as the number of runs it has facilitated. The loss in transition can be represented as the number of lost runs per transition. This is captured in the following matrix, where *DM* stands for degradation matrix and *lr* for lost runs. Assuming all color grades of stone cause the same wear and tear on the mold, the diagonal elements are 1 signifying no transition. A transition from grade 1 to 2 would cause lr_{12} >1 number of lost runs.

$$DM = \begin{vmatrix} 1 & lr_{12} \ldots lr_{1n_p} \\ lr_{21} & 1 \ \ldots lr_{2n_p} \\ \ldots\ldots & \ldots\ldots \\ lr_{n_p1} & lr_{n_p2} \ldots . 1 \end{vmatrix}$$

For modeling purposes, it is assumed that when a new mold enters the system, it has a finite life θ_T associated with it. This implies that the mold would deliver θ_T runs before breaking. θ_T can be constant, deterministic or drawn from a distribution leading to stochastic resource ageing in the latter case.

3.2.2. Demand modeling

Demands for different types of products are often mutually or temporally correlated. While the mutual correlation is neglected in this study, the temporal correlation is addressed by considering the following two cases:

(i) Cyclical demand – In this situation, demand for products follows a repeating trend as shown in Figure 3.2(a). If one demand scenario represents the set of demands of n_P products at a given time, the scenario repeats every r time periods. E.g., if we have a single product (n_P =1) and r=5, then a cyclical demand pattern can be represented as in Figure 3.2(b), where the circles show the demand for product and the arrows show the transition from one scenario to the other over one time period. In section 3.4.1, this concept is extended to include random transitions from one scenario to the other. The model is commonly known as a Markov chain.

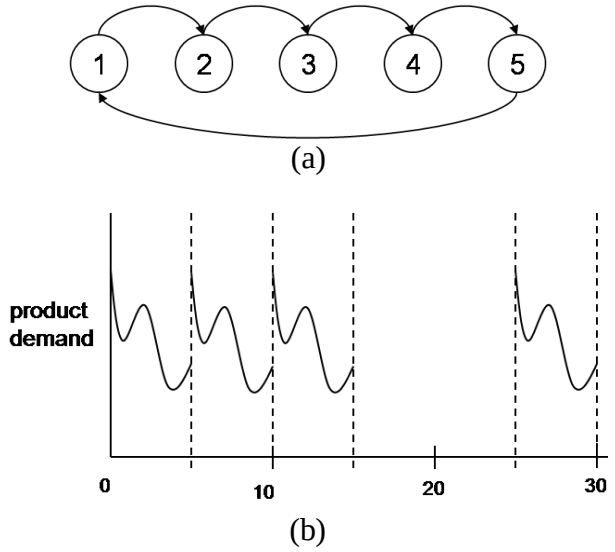


Figure 3.2: Demand patterns (a) cyclical demand pattern. (b) Cyclical demand pattern over a 30 unit horizon

(ii) Seasonal demand – In this case, the demand for products is low, medium or high depending on the time of the year. The demand trend is illustrated by an example with n_P =1 in Figure 3.3. Again, the stochastic elements are introduced in section 3.4.

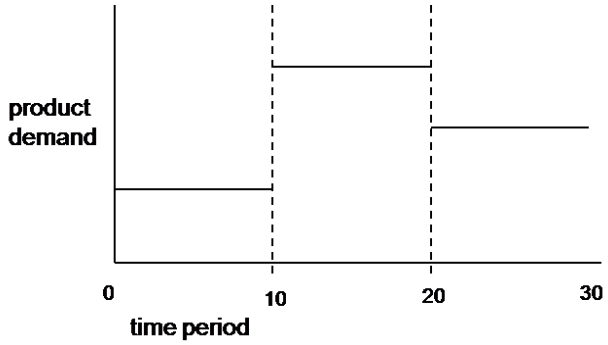


Figure 3.3: A seasonal demand scenario

3.2.3 Resource replenishment lead time and reorder limit

It is assumed that there is a lead time associated with the delivery of new resources. This implies that new resources are ordered during the current time period, they will arrive after *l* time periods, where *l* is the replenishment lead time. The lead time is treated as a deterministic parameter in all subsequent analyses. Additionally, there is an upper limit on reorder quantities of the molds. The restriction is applied to keep the reorder quantities from fluctuating significantly.

The plant capacity is assumed to be 20 runs per day for all subsequent analysis. As shown in Figure 3.4, the scheduling time period consists of one day and the scheduling time horizon *h* is 10 days. Alternatively, planning time period is consists of 10 days and planning is done over a horizon *H* of 30 time periods. Due to difference in time scales, the decision-making is divided between the levels as outlined in section 3.1. At the planning level, the optimal reorder quantities of the resource and the desired ending

inventory of products and resources at the end of each time period are determined. These values are then communicated to the scheduling level to generate optimal production schedule while also allocating the resources optimally. The details of formulation are presented in the next section. To gain an insight into the trade-offs involved, the deterministic problem is considered first.

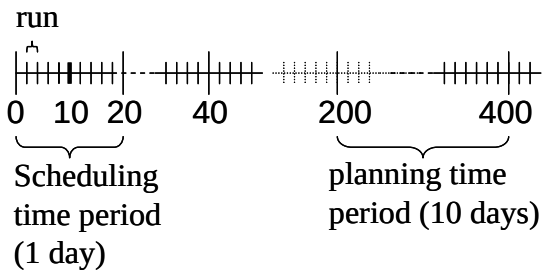


Figure 3.4: Time scales associated with the planning and scheduling problems considered in this work.

3.3. Deterministic problem –LP formulation

3.3.1 Deterministic demand and resource life

When the demand for products and number of runs that a resource delivers, have no associated uncertainty, the problem can be formulated as a linear program (LP) at the planning level. The planning LP generates optimal production and resource inventory at the end of each planning period, while satisfying production capacity constraints. Due to absence of uncertainty, the scheduling level can completely satisfy the inventory targets prescribed by the solution of planning LP. It is assumed that new orders for resources can be placed only at the beginning of each planning period and product demand is satisfied only at the end of planning periods. The detailed formulation is presented below.

Planning LP – model M1

The objective at the planning level is to maximize profit from product sales over the entire planning horizon. Production and resource reorder costs are accounted for, and constraints related to plant capacity and resource availability are applied. Holding costs for excess inventory of products and resources are also incurred. It should be noted that excess inventory of products and resources is needed even when the demand is deterministic but fluctuates significantly. In order to stabilize production and/or avoid capacity issues, inventory is built up at times of lower demand in anticipation of high periods in future. Finally, so as not to drain the inventory at the end of planning horizon, ending inventory for products and resources is set to be equal to the inventory values at the beginning of the horizon. The objective is achieved by making optimal decisions about production, resource transition and the number of new resources to bring into the system during each planning period. The associated variables are inventory of products and resources, product sales and resources that will break at the end of each planning period.

As a practical requirement, the transition of resources is performed in a uni-directional fashion. The set i of different types of products is ordered so that only the transitions from $i1$ to $i2$ such that $i1<i2$ are allowed for $i1 \in i$. It is also assumed that the new resources are ordered only for $i1=1$. E.g., if there are two types of products {light,dark} and only transitions between light to dark are allowed, then the new resources are always dedicated to making light products. They can be transitioned to make dark products in the course of time. Also, when the new resources arrive, their ages are assumed to follow a deterministic pattern. In section 3.4, where stochastic resource life is considered, the terminal age is modeled as a normal distribution. Therefore a deterministic pattern similar to normal distribution is chosen. This is shown in Figure 3.5. E.g. if a total of 10 new resources are ordered and the terminal resource life for the resources is distributed around a mean age of 17 runs, then 30% of the resources will have an age of 17, 20% will have ages 16 and 18, 10% will have ages 15 and 19 and 5% will have ages 14 and 20. This assignment is achieved using vector A_j in the planning LP model where the vector is given by [0 0 0 0 0 0 0 0 0 0 0 0 0 5 10 20 30 20 10 5].

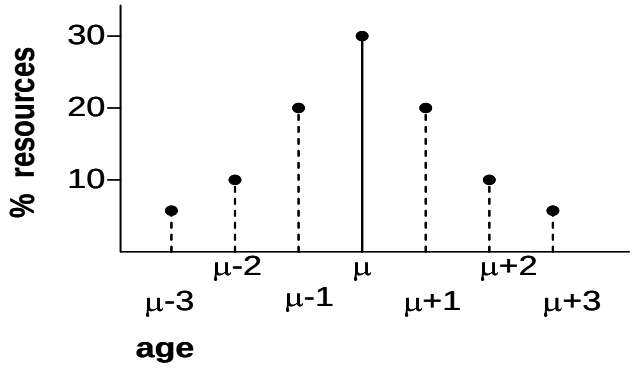


Figure 3.5: Deterministic initial age distribution of new resources

In the planning LP (model M1), profit is given by the difference of revenue from product sales and cost of production, resource reorder and holding inventory of products and resources. Equations *M*1.1 through *M*1.4 represent the mass balance of products and resources. The resources have two associated attributes; the type of product (*i*) they are dedicated to making and the number of runs remaining (*j*). While the mass balance equations for products are straightforward, those for the resources include the new arrivals, the degradation of resources due to use ($y3_{ij,j1,t}$) and the transition of resources from one product type to another ($y2_{i,i1,jt}$). The degradation term $y3_{ij,j1,t}$ represents the usage of resources during a planning period. Since the planning period has a length *h* and a resource can be used only once a day, it cannot be used more than *h* times. Therefore, the difference *j*-*j*1 must be less than *h*. It is assumed that no runs are lost in the transition operation.

In the presence of replenishment lead time *l* for new resources, information about previously placed orders that are scheduled to arrive at time *t* ($y1^{0}_{ijt}$) needs to be provided (equation *M*1.3). Similarly, the re-order decision variable $y1_{ijt}$ has an associated lag *t*-*l* as shown in equation *M*1.4.Equation *M*1.5 represents the fact that product sales cannot exceed demand and *M*1.6 through *M*1.7 place a limit on production in the form of plant capacity and resource availability respectively. *M*1.8 assigns ages to newly arrived

resource according to assignment vector A_j. Finally $M1.9$ and $M1.10$ are hard constraints on the inventory of products and resources at the end of the planning horizon and $M1.11$ are the positivity constraints.

Model M1

$$z1 = \max \sum_t \sum_i R_i x3_{it} - \{\sum_t \sum_i P_i x1_{it} + \sum_t \sum_i \sum_j E_i y1_{it} + \sum_t \sum_t H_i x_{it} + \sum_t \sum_i \sum_j H_{ij} y_{ijt}\}$$

s.t.

$$x_{it} = x_i^0 + x1_{it} - x3_{it} \qquad \forall i, t = 1 \qquad (M1.1)$$

$$x_{it} = x_{i,t-1} + x1_{it} - x3_{it} \qquad \forall i, t = 2,3,...H \qquad (M1.2)$$

$$y_{ijt} = y_{ij}^0 + y1_{ijt}^0 + \sum_{i1<i} y2_{i1,i,jt} - \sum_{i1>i} y2_{i,i1,jt} + \sum_{j1=j+1}^{j+h} y3_{ij1,jt} - \sum_{j1=j-1}^{1} y3_{ij,j1t} \qquad \forall i, \forall j, t \le l \qquad (M1.3)$$

$$y_{ijt} = y_{ij,t-1} + \bar{y}1_{ij,t-l} + \sum_{i1<i} y2_{i1,ijt} - \sum_{i1>i} y2_{i,i1,jt} + \sum_{j1=j+1}^{j+h} y3_{ij1,j,t} - \sum_{j1=j-1}^{1} y3_{ij,j1,t} \qquad \forall i, \forall j, t > l \qquad (M1.4)$$

$$x3_{it} \le D_{it} \qquad \forall i \qquad (M1.5)$$

$$x1_{it} \le \sum_j \sum_{j1} (j - j1) * y3_{ij,j1,t} \qquad \forall i, t, j > j1 \qquad (M1.6)$$

$$\sum_i x1_{it} \le h * C \qquad \forall t \qquad (M1.7)$$

$$\bar{y}_{ijt} = A_j y_{it} \qquad \forall t \qquad (M1.8)$$

$$x_{iH} \ge x_i^0 \qquad \forall i \qquad (M1.9)$$

$$y_{ijH} \ge y_{ij}^0 + y1_{ij,1}^0 \qquad \forall i, j \qquad (M1.10)$$

$$x_{it}, x1_{it}, x3_{it}, y_{ijt}, y1_{ijt}, y2_{i,i1,jt}, y3_{i,j,j1,t} \ge 0 \qquad \forall i, j, t \qquad (M1.11)$$

The deterministic problem is solved for a number of cases with varying parameter values for product demand and replenishment lead time. The different cases are summarized in the Table 3.1. Discussion on the results is deferred until section 3.6.

3.4. Dealing with stochastic problems

In most real world situations, parameters like product demand, resource life and resource replenishment lead time are not deterministic. The concepts and models developed in the previous section therefore need to be modified. The replenishment lead time of new resources is considered deterministic in this study since it is possible to regulate the delivery of the resources by supplier management. Product demand and resource life are modeled as stochastic parameters as shown below.

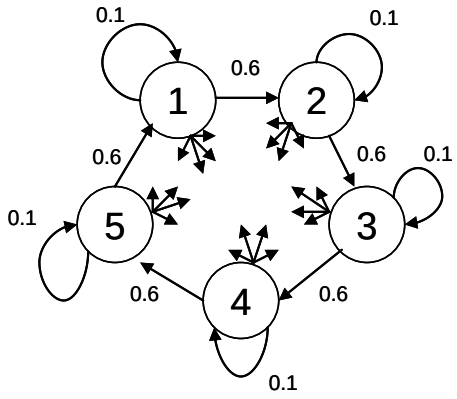
3.4.1 Stochastic demand modeling

The stochastic demand in this study is modeled as an extension to the previously considered demand patterns in section 3.2.2.

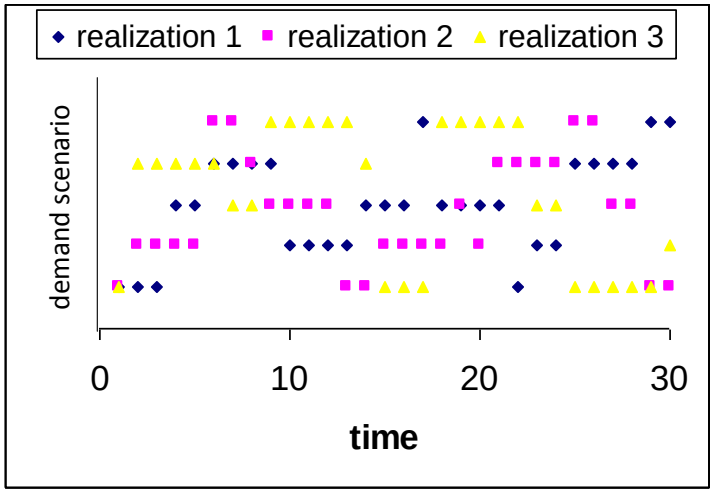
(i) Highly fluctuating demand – On similar lines as the cyclical demand pattern, the demand is modeled as a Markov chain shown in Figure 3.6(a). Each circle represents a demand scenario, i.e., the set of demands for all products at a given instant. The arrows emanating from the circle and connecting it to other circles represent the possible scenarios at the next time period. One of the possible scenarios is realized at the next time period depending on the probability associated with the arrow. Starting with a unique demand scenario, the demand can follow many trajectories with varying probabilities as shown in Figure 3.6(b). Since it no longer follows a cyclical trend, the pattern is called highly fluctuating demand.

(ii) Seasonal demand - Instead of considering high, low and medium levels of demand shown in Figure 3.7(a), the demand is modeled to lie within a band during each of the three parts of the year. The demand transitions deterministically from a low to high, and high to medium period, but the

transitions within the band are stochastic. Three equi-probable values within the band are assumed at each time period. This would again result in numerous demand trajectories with associated probabilities, some of which are shown in Figure 3.7(b).

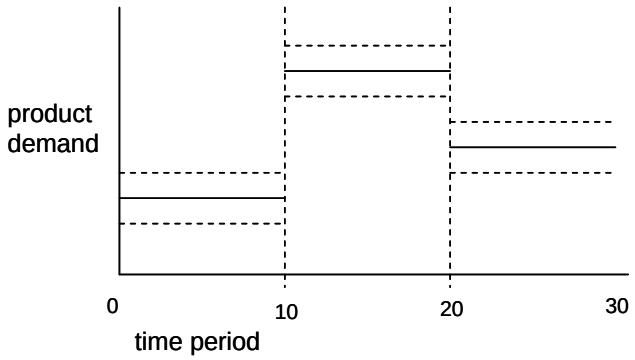


(a)

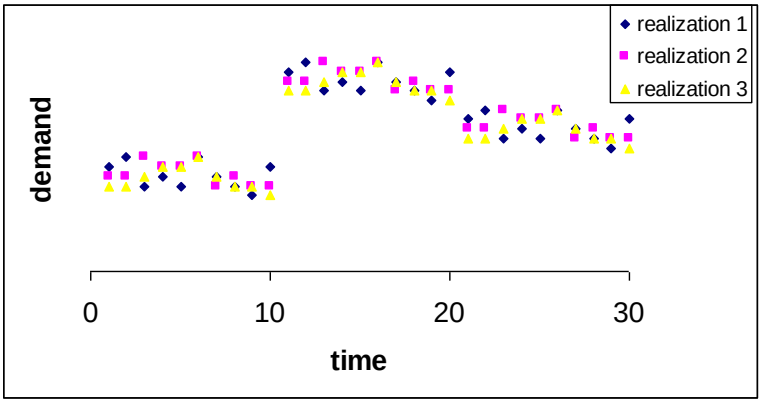


(b)

Figure 3.6: (a) Illustration of highly fluctuating stochastic demand pattern. (b) Realizations of a highly fluctuating demand pattern.



(a)



(b)

Figure 3.7: (a) Seasonal demand pattern prone to uncertainty (b) Realizations of the stochastic seasonal demand pattern

3.4.2 Modeling stochastic resource life

As mentioned in section 3.2, a new resource is assumed to enter the system with a terminal age θ_T, which is essentially the number of runs that the resource will deliver throughout its life. In section 3.3, θ_T was assumed to be following a deterministic pattern similar to a normal distribution. In this section, θ_T is assumed to be normally distributed with mean μ and standard deviation σ. This implies that if 15 new resources entered the

system at time t and $\theta_T \in \aleph(17,2)$ then a possible realization of resource life distribution is [0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 3 4 4 0 2 1] and several such realizations are possible.

3.4.3 Solving the stochastic problem

In usual practice, the planning problem is solved for projected future demand for products. The optimal ending inventory for products and resources is then generated using the LP described in previous section. Therefore, the desired inventory at the end of each 10 day period is known. During these 10 days, the scheduling level generates a detailed production schedule and resource allocation schedule to meet the ending inventory requirement at the end of the 10 day period. However, when resource durability and product demand have associated uncertainty, the ending inventory values of products and the resource life profile are not the same as predicted by the planning LP. The actual inventory values and resource life distribution need to be communicated to the planning LP at the end of each planning period. This is achieved by solving the LP in a rolling horizon fashion as shown in Figure 3.8. At the beginning of the horizon, the LP is solved for 30 time periods and the ending inventory targets for products and resources are generated. The decisions about resource reorder, production etc. are implemented at the beginning of the horizon. At the end of first time period, the age distribution of the new resources and the actual product demand become available. The horizon is shifted by one time period and the LP is solved again for next 30 time periods starting with the actual values of product inventory and resource life distribution.

Moreover, the ending inventory targets prescribed by the planning level may not be feasible at the scheduling level. E.g., if the demand for products at the end of last planning period was substantially higher than the projected value, the system is left with lower inventories at the beginning of the current period. Production capacity and resource limits may not allow enough production for the scheduling level to match the ending inventory target. Alternatively, if the resource lives are realized to be lower than expected, then the resource availability may not be enough to produce goods up to the prescribed inventory targets. Consequently, an optimization problem at the scheduling

level needs to be solved taking into account the actual starting inventory and actual number of runs available from the resources at the beginning of each planning period. The time horizon *h* for the scheduling period is 10 days (equal to one planning period) and one scheduling time period is one day. The difference in time scales of planning and scheduling are shown in Figure 3.4 and the scheduling LP is shown by model M2.

The scheduling LP has similar mass balance equations, capacity constraints, resource allocation and transition variables as contained in the planning LP. The production, inventory, transition and allocation variables in this case are designated with a tilda (~) sign to differentiate from those at the planning level. The important differences from the planning LP being:

(i) Resource reorder and product sales are not a part of the decision-making

(ii) Since revenue is not being generated at this level, the objective is to minimize cost of production while meeting the ending inventory targets at the end of the scheduling horizon (10 days) as prescribed by the planning LP. The latter is modeled as penalty variables $penalty1_i$, $penalty2_i$ as shown in the model. If the system ends up with less inventory of product *i*, i=1,2,..n_P and less number of remaining runs to make product *i*, i=1,2.. n_P (equations *M*2.7 and *M*2.8), a cost proportional to the difference is added in the objective function. Weights *w*1 and *w*2 are reasonably chosen parameters.

(iii) As per the assumptions stated in section 3.2, the resource delivers one run in one scheduling period as shown using allocation variables $y3_{ijt}$ in equations *M*2.3 and *M*2.4.

(iv) The total production in one scheduling period cannot exceed the system capacity of *C* runs/day as shown by capacity constraint *M*2.6.

(v) The ending inventory of products ($\tilde{x}_{ih}$) and number of runs remaining from the resources ($j * \tilde{y}_{ijh}$) are targeted to be more than or equal to those prescribed by the planning LP (x_{i1}, $j * y_{ij1}$). This is accomplished by constraints *M*2.7 and *M*2.8 as described in point (ii).

Model M2

$$z2 = \min \sum_t \sum_i P_i \tilde{x}1_{it} + w1 \sum_i penalty1_i + w2 \sum_i penalty2_i$$

s.t.

$$\tilde{x}_{it} = x_i^0 + \tilde{x}1_{it} \qquad \forall i, t = 1 \qquad (M2.1)$$

$$\tilde{x}_{it} = \tilde{x}_{i,t-1} + \tilde{x}1_{it} \qquad \forall i, t = 2,3,...h \qquad (M2.2)$$

$$\tilde{y}_{ijt} = y_{ij}^0 + \sum_{i1<i} \tilde{y}2_{i1,ijt} - \sum_{i1>i} \tilde{y}2_{i,i1\,jt} + \tilde{y}3_{i,j+1,t} - \tilde{y}3_{ijt} \qquad \forall i, \forall j, t = 1 \qquad (M2.3)$$

$$\tilde{y}_{ijt} = \tilde{y}_{ij,t-1} + \sum_{i1<i} \tilde{y}2_{i1,ijt} - \sum_{i1>i} \tilde{y}2_{i,i1\,jt} + \tilde{y}3_{i,j+1,t} - \tilde{y}3_{ijt} \qquad \forall i, \forall j, t = 2,3.,....h \qquad (M2.4)$$

$$\tilde{x}1_{it} = \sum_j \tilde{y}3_{ijt} \qquad \forall i, t \qquad (M2.5)$$

$$\sum_i \tilde{x}_{it} \le C \qquad \forall t \qquad (M2.6)$$

$$x_{i1} - \tilde{x}_{ih} \le penalty1_i \qquad \forall i \qquad (M2.7)$$

$$\sum_j \{ j * y_{ij1} - j * \tilde{y}_{ijh} \} \le penalty2_i \qquad \forall i, j \qquad (M2.8)$$

$$\tilde{x}_{it}, \tilde{x}1_{it}, \tilde{x}3_{it}, \tilde{y}_{ijt}, \tilde{y}1_{ijt}, \tilde{y}2_{i,i1,jt}, \tilde{y}3_{i,j,j1,t} \ge 0 \qquad \forall i, j, t \qquad (M2.9)$$

3.4.4 Rolling horizon approach

In order to evaluate the actual performance of the stochastic system, a simulation model is employed. The simulation model combines the process of taking realizations of parameter values at the end of each planning period with the optimization at planning and scheduling levels in a rolling horizon fashion. The schematic for the model is shown in

Figure 3.9. Variables x_i^t and y_{ij}^t (with subscript *t*) denote the actual inventory of products and resources at the end of planning time period *t*. A stochastic simulator is employed to generate instances of the random parameters (product demand and resource life). At the beginning of the simulation model, *t*=1 and the starting inventory values are given by x_i^0 and y_{ij}^0. The planning LP is solved for x_i^t, y_{ij}^t and $y1_{ij}^t$ for i=1,2,..n_P; j=1,2,…θ_T and *t*=1,2..30, where the variables stand for target product inventory, target resource inventory and resource reorder decision respectively at the end of each of the 30 time periods. Input in the form of the product demand at the end of first planning time period and new resources scheduled to arrive throughout the planning horizon is provided. The target inventory of products and resource at the end of the first period is fed as input to the scheduling LP, while the reorder decision is fed to the random number generator. The latter generates a random life distribution for the new resources $Y1_{ijt}$ according to a pre-specified normal distribution. $Y1_{ijt}$ is the actual inventory of new resources scheduled to arrive after a lead time of *l* periods. The scheduling LP uses as inputs, the actual starting inventory values (x^0,y^0) and the target inventory values (x_i^1 and y_{ij}^1) passed by the planning level, to generate optimal production, resource allocation and transition scheme for one scheduling horizon (10 days). The target inventory for products from planning period indicates the inventory before product sales. Therefore, *x*3 is added to the target. Since scheduling problem is completely deterministic, the inventory values at the end of the scheduling horizon *h* (x_h,y_h) are the actual values. At the end of the first planning period, the demand for products is realized. Also, the new resources that arrive at the end of this time period are added to the resource inventory. The resulting inventory values are $x_i^1 = \max(x_{ih} - D_{i1}, 0)$ and $y_{ij}^1 = y_{ijh} + Y1_{ijh}$. This completes one simulation run. The time counter is shifted by one time period and the run is repeated until *t*=30.

The LP at planning level is solved using CPLEX solver through GAMS (version 19.6) and the simulation is performed in Matlab (version 7a). The interfacing software by Ferris (1998) is used to perform the optimization and simulation repeatedly.

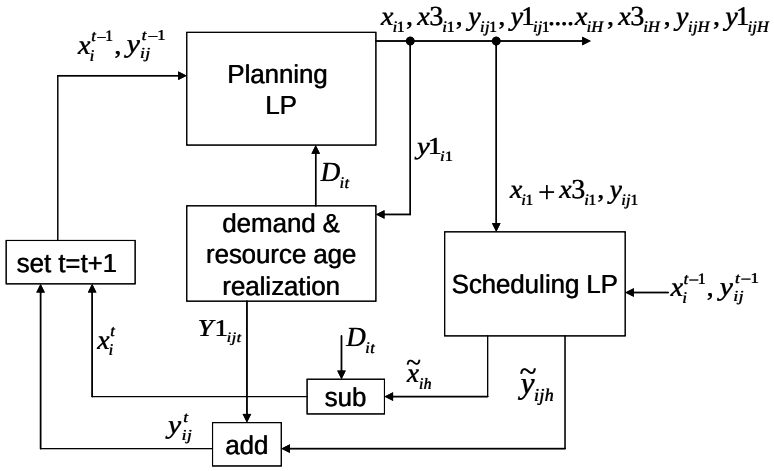


Figure 3.8: Flow diagram for the rolling horizon solution approach

1. Set k=1

   Initialize x_i^0 =0, y_{ij}^0 = 0, $y1_{ijt}^0$ with values based on lead time l

2. Use stochastic simulator to obtain D_i^k , the demand at the end of time period k

3. Solve M1 (planning LP) to obtain $x_{it}, y_{ijt}, y1_{it}$ for i=1,2,…n_P, j=1,2..20 , t=1,2,…30

4. use $y1_{i1}$ to obtain $Y1_{ijt}$ using stochastic simulator

5. Solve M2 (scheduling LP) using $x_{i1} + x3_{i1}, y_{ij1}, x_i^{k-1}, y_{ij}^{k-1}$ to obtain $\tilde{x}_{it}, \tilde{y}_{ijt}$ for i=1,2,.. h

6. Obtain $x_i^k = \max(\tilde{x}_{ih} - D_i^k, 0)$

   and $y_{ij}^k = \tilde{y}_{ih} + Y1_{ijk}$

7. If k=30, stop

   else set k=k+1 and go to step 2.

Figure 3.9: Solution algorithm for the rolling horizon solution approach.

An important thing to note is that when the deterministic demand forecast for the next H time periods is fed into the planning LP, a buffer needs to be maintained so that the fluctuations in future demand are accounted for. A conservative heuristic is employed to this end, by inclusion of a pre-determined buffer stock or safety stock *ss*, where *ss* is treated as a parameter to be optimized. E.g. 10% extra is added to the forecast. Any excess is automatically corrected at the next time period.

The rolling horizon approach is conventionally applied to planning and scheduling problems for production planning independent of resource planning which is done separately. This is shown in the next section.

3.5. The decoupled problem

In order to gauge the effectiveness of solving the coupled problem, the decoupled problem is also solved. This implies that the resource reorder and production planning decisions are considered independent of each other (conventional practice).

3.5.1 Resource reorder

Resource procurement, when treated independent of production planning, is a standard inventory control problem with stochastic demand. In such a scenario, a policy of the form (*s*,*S*) is proven optimal for given demand distributions and provides a good approximation for the others. The parameter *s* is the reorder point and *S* is the reorder quantity. The optimal plan is to place an order whenever the stock is equal to or falls below *s* to bring the inventory to level *S* and do nothing otherwise. Since there is no fixed cost associated with placing a new order in the system considered, *s* is very high. The policy in this case is to always bring the stock of resources up to level *S*, where *S* is the only parameter to be optimized. Taking note of the flexibility of the resources and the fact that the resources deliver multiple runs, different quantities may be considered as stock in this case. E.g.,

(i) Total number of resources at any time

(ii) Separate parameters for different types of resources (S_i for i=1,2,…n_P)

(iii) Total number of runs available from all the resources.

(iv) Total number of runs available from each type of resource.

For this analysis, total number of resources is considered as stock.

3.5.2. Production planning

The optimal production quantities need to be determined at the planning level, in the face of uncertain demand. Since resource procurement is no longer a part of decision-making, only production variable ($x1_i$), product inventory (x_i) and product sale ($x3_i$) are a part of the planning LP. The model M1 is now modified to M3 with constraints related to product mass balance, demand and supply and system capacity. Resource availability is assumed to be 100% at all times.

Model M3

$$z3 = \max \sum_t \sum_i R_i x3_{it} - \{\sum_t \sum_i P_i x1_{it} + \sum_t \sum_i H_i x_{it}\}$$

s.t.

$$x_{it} = x_i^0 + x1_{it} - x3_{it} \qquad \forall i, t = 1 \qquad (M3.1)$$

$$x_{it} = x_{i,t-1} + x1_{it} - x3_{it} \qquad \forall i, t = 2,3,...H \qquad (M3.2)$$

$$x3_{it} \le D_{it} \qquad \forall i \qquad (M3.3)$$

$$\sum_i x1_{it} \le h * C \qquad \forall t \qquad (M3.4)$$

$$x_{iH} \ge x_i^0 \qquad \forall i \qquad (M3.5)$$

$$x_{it}, x1_{it}, x3_{it} \ge 0 \qquad \forall i, j, t \qquad (M3.6)$$

Having obtained the planning level decisions, as shown above, scheduling model (M2) is solved in a rolling horizon fashion. The difference is that the two decisions (resource procurement and production planning) come from separate models in this case. The method is outlined in Figure 3.10.

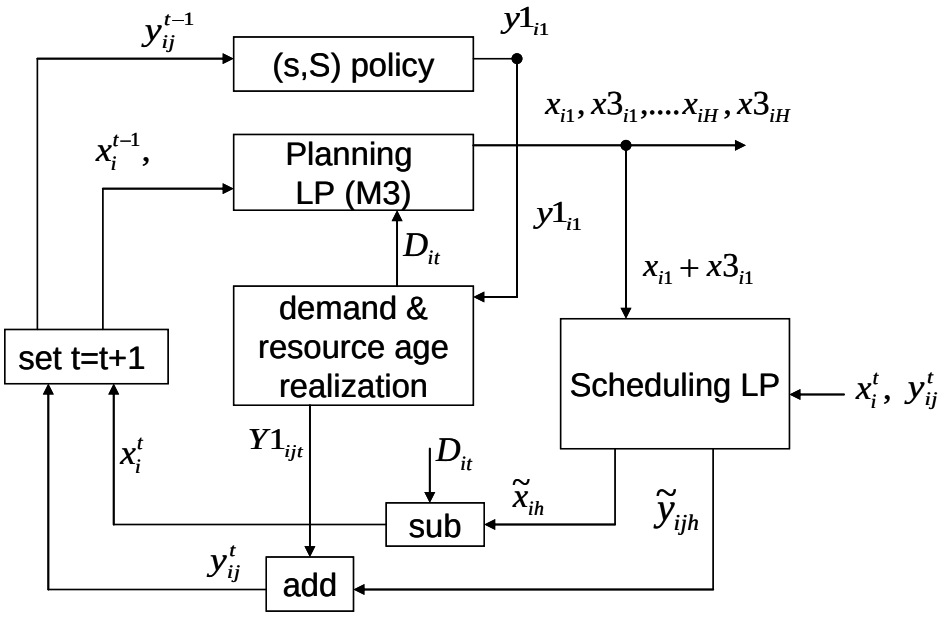


Figure 3.10: Flow diagram for the solution of the decoupled problem

This approach proves advantageous in terms of computation time since the decision variables corresponding to the high dimensional resource vectors are dropped from optimization, but the simplification comes at the cost of potentially significant loss in solution quality. A discussion on the performance of this solution approach and comparison with that of the others is presented in the ensuing section. For simplicity, the solution approach described in section 3.3 is referred to as approach I, and those in sections 3.4 and 3.5 as approach II and III respectively.

3.6. Numerical results and discussion

3.6.1 Parameter values

The solution approaches I,II and III are tested on the example with cost parameters shown in Table 3.1. The system consists of 2 types of products namely light and dark (n_P=2). Allowable resource transitions are from light to dark only.

Table 3.1: Parameter values for the perishable resource problem

n_P	2	R_i	500 $\forall i$	H_{ij}	0.00002 $\forall i, j$
C	200	P_i	45 $\forall i$	w_1	1000
H	30	E_i	2000 $\forall i$	w_2	10
h	10	H_i	0.002 $\forall i$		

The nominal resource life is 17 runs and the nominal demand scenarios for the constant, cyclical and seasonal cases are given by $d^{con}, d^{cyc}, d^{sea}$ respectively for the two types of products.

$$d^{con}(t) = [100\ 100] \quad for\ i = 1,2;\ t = 1,2,...,30$$

$$d^{cyc}(t) = \begin{cases} [80\ 80] & if\ \mathrm{mod}(t,5) = 1 \\ [80\ 125] & if\ \mathrm{mod}(t,5) = 2 \\ [125\ 125] & if\ \mathrm{mod}(t,5) = 3 \\ [125\ 80] & if\ \mathrm{mod}(t,5) = 4 \\ [110\ 110] & if\ \mathrm{mod}(t,5) = 0 \end{cases} \quad for\ t = 1,2,..30$$

$$d^{sea}(t) = \begin{cases} [60\ 60] & if\ \ 1 \le t \le 10 \\ [140\ 140] & if\ \ 11 \le t \le 20 \\ [100\ 100] & if\ \ 21 \le t \le 30 \end{cases}$$

The uncertainty is modeled as discussed in section 3.4. In the case of fluctuating demand, the scenarios are chosen so that the demand fluctuates around a constant mean of [100 100] for both product types, during one planning period. High level of uncertainty signifies higher deviations from the mean.

It is also assumed that once a scenario is realized, the probability of the same scenario being realized at the next instant dominates over all others. This ensures that the demand stays at a given value for longer times, as seen in reality. Twelve deviations from nominal demand are considered as scenarios sets *s*=1,2,..12 shown in Table 3.2. L and D correspond to light and dark products respectively. The probabilities of transition among these scenarios govern the uncertainty, where higher deviations from mean demand signify higher level of uncertainty. This is captured by three transition

probability matrices $T1$, $T2$ and $T3$ in the order of increasing level of uncertainty. s′ denotes the scenario at the next time period and all matrices are diagonally dominant.

Table 3.2: Deviations from nominal demands for the Light and Dark products

s	L	D
1	0%	0%
2	-25%	10%
3	10%	-25%
4	20%	20%
5	-60%	40%
6	40%	-60%
7	-40%	60%
8	60%	-40%
9	45%	45%
10	-40%	-40%
11	-70%	-70%
12	65%	65%

$$T1(s,s') = \begin{cases} 0.8 & if\ \ s = s' and\ s' \le 4 \\ \dfrac{0.2}{3} & if\ \ s \ne s'\ and\ \ s' \le 4 \\ 0 & otherwise \end{cases} \qquad T2(s,s') = \begin{cases} 0.8 & if\ \ s = s' and\ s' \le 8 \\ \dfrac{0.2}{7} & if\ \ s \ne s'\ and\ \ s' \le 8 \\ 0 & otherwise \end{cases}$$

$$T3(s,s') = \begin{cases} 0.8 & if\ \ s = s' \\ \dfrac{0.2}{11} & otherwise \end{cases}$$

In order to account for seasonality in demand, the same levels of uncertainty are considered for the seasonal nominal case. Multiple realizations for each type of demand

(constant and seasonal mean) and each level of uncertainty (low, medium and high) are considered to obtain average performance. (It must be noted that there are 12^{30} possible realizations, making the probability of an individual realization miniscule. Therefore obtaining a true average is very difficult). The realizations are drawn at random and are fixed for the different solution approaches. The comparison is mostly scenario based. Resource life is modeled as a normal distribution with mean of 17 runs. Low, medium and high levels of uncertainty correspond to standard deviations of 0.5, 1 and 2 respectively.

3.6.2 Features

The models and solution approaches described throughout this article are tested along the features listed in Table 3.3, where lead time is the delay associated with resource order arrival, safety stock is the buffer added to the demand forecast. Two types of demands and three levels of uncertainty are considered as discussed. Various revenue and cost heads are reported in Table 3.4 for some of the cases. Average resource utilization defined as the ratio of total number of resources used and the total number of resources in the system is also reported along with the total missed demand to highlight the instances of lost opportunity. The effects of aforementioned features are examined after some general observations outlined below:

(i) Product sales contribute highest to the overall profit. Therefore, satisfying product demand is the chief driver for all solution approaches. The higher the missed demand, the worse is the performance.

(ii) The optimal issuing policy is to use the oldest resources and optimal transition policy is from oldest light resource to dark.

(iii) Different demand trajectories sum to different values of overall demand. Therefore, profit is scaled by the overall demand as the objective value in all analyses.

Table 3.3: Problem features and their levels considered for analysis

Parameter level	**I**	**II**	**III**
Parameter name			
Demand type	Constant	fluctuating (cyclical)	seasonal
Uncertainty	Low	medium	high
Lead time in mold order arrival	1 unit	3 unit	5 units
Safety stock (as % of product demand)	5	10	20
Solution approach	Deterministic LP	Rolling Horizon	Heuristic (Q,r) for mold reorder

Table 3.4: Performance of solution approaches I,II and III for various levels of uncertainty.

Lead time in mold arrival is 3 in all the cases. (ss – safety stock)

	solution method	uncertainty level	total sales	total production	total reorder	avg product inventory	avg resource inventory	avg resource utilization	total missed demand
deterministic									
constant	I	-	6000	6000	341.57	29.86	31.93	84.42%	0.00
cyclical	I	-	6000	6000	341.00	109.82	32.01	86.27%	0.00
seasonal	I	-	6000	6000	340.24	292.70	32.72	85.27%	0.00
stochastic									
constant	I	L	4438.8	4418.0	301.57	16.78	22.26	89.06%	1721.22
	I	M	4264.8	4284.8	301.57	42.61	22.41	88.55%	1955.20
	I	H	4434.6	4404.6	301.57	28.04	22.65	89.48%	2465.40
seasonal	I	L	4430.2	4400.2	300.24	99.26	21.80	91.29%	1735.78
	I	M	4361.5	4333.7	300.24	139.85	22.03	90.43%	1970.48
	I	H	4337.2	4307.2	300.24	65.34	21.52	90.85%	2046.83
constant (ss- 20%)	II	L	5352.7	5359.7	320.85	35.86	25.94	79.17%	807.27
	II	M	5268.5	5250.5	309.13	27.50	26.29	74.55%	951.54
	II	H	5209.0	5203.0	311.31	35.76	26.48	74.19%	1691.00
seasonal (ss- 20%)	II	L	4963.8	4970.8	284.58	40.44	26.56	73.39%	1202.20
	II	M	4618.4	4600.4	267.14	29.65	26.21	69.03%	1713.53
	II	H	4569.6	4563.6	287.08	35.63	23.63	68.92%	1814.40
constant	III	L	2822.0	2822.0	256.00	8.00	18.06	75.36%	3338.00
	III	M	2670.6	2640.6	261.00	4.67	16.38	74.85%	3549.40
	III	H	2901.9	2871.9	250.69	15.73	18.04	78.59%	3998.10
seasonal	III	L	3032.2	3002.2	263.50	32.08	19.04	77.85%	3133.80
	III	M	2938.6	2908.6	258.90	28.22	17.95	82.16%	3393.40
	III	H	2620.5	2590.5	245.09	5.63	16.21	79.85%	3763.50

The effects of different features are now discussed:

A. Effect of time varying demand

To analyze the effect of time varying demand, the deterministic demand and resource life scenarios are studied. For three types of demand (constant, cyclical and seasonal), substantial difference lies in the values of average product inventories as seen in Figure 3.11(a). This is because when demand is uneven, inventory is built up beforehand. Nominal seasonal demand has higher deviations from the constant trajectory and therefore has higher average inventory of products. As a result, the production is leveled, as seen by the similar average resource inventories. The other revenue and cost heads are the same because although the demand deviates, it is known completely to the decision-maker.

B. Effect of uncertainty

In most cases, higher uncertainty results in lower performance in terms of profit per unit of demand for both approaches (I and II) as shown in Figure 3.11(b). This is also reflected in increasing values of missed demand. It is seen that although the resource reorder quantities are similar to the deterministic solution (imposed to be equal for approach I), the average resource inventories in both cases are lower. This signifies more resource breakage due to uncertainty in resource lives.

C. Lead time

Increase in lead time results in worse performance by both solution approaches (I and II) as seen in Figure 3.11(c). The result is expected because the further out into the future we look, the more uncertain demand gets.

D. Safety stock

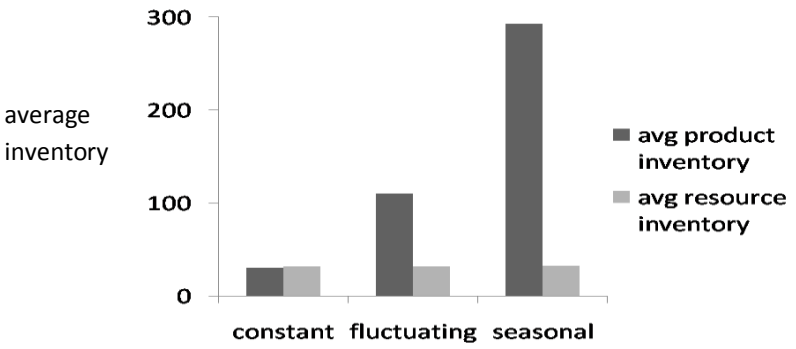
Four values of safety stock or demand buffer are considered for approach II. In both demand profiles, 20% safety stock is the best heuristic (Figure 3.11(d)). As safety stock is increased further, the system keeps more inventory of products than required, and

one product cannibalizes the other in terms of its share of resources and production capacity.

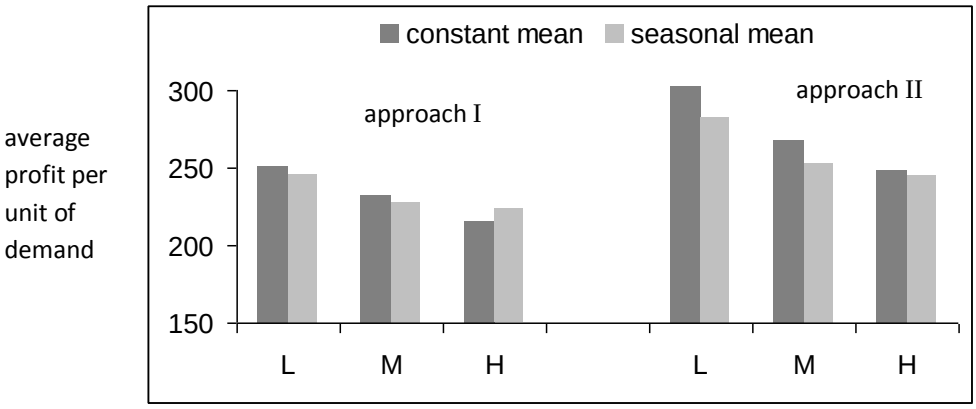
E. Solution approaches

A comparison of the performance of the three approaches (I,II and III) is shown in Figure 3.11 (e) for the two types of demand (constant nominal demand and seasonal nominal demand). The decoupled problem (approach III), is solved for several values of S (30,35,40,45,50), where *S* is the maximum stock level. The best value for *S* is found to be 35 and the results are reported for this value only. It is seen that decoupled problem in all cases performs significantly worse than the LP without update and the rolling horizon approach. As much as 50% demand is missed in certain cases when using approach III. This confirms that treating resource procurement and production planning decisions together is imperative. The decoupled problem would perform particularly badly when the total resource usage varies significantly from one period to the other. The latter is likely in most practical situations when either the product demand deviates significantly from the mean or resource allocation causes varying number of resources to perish.

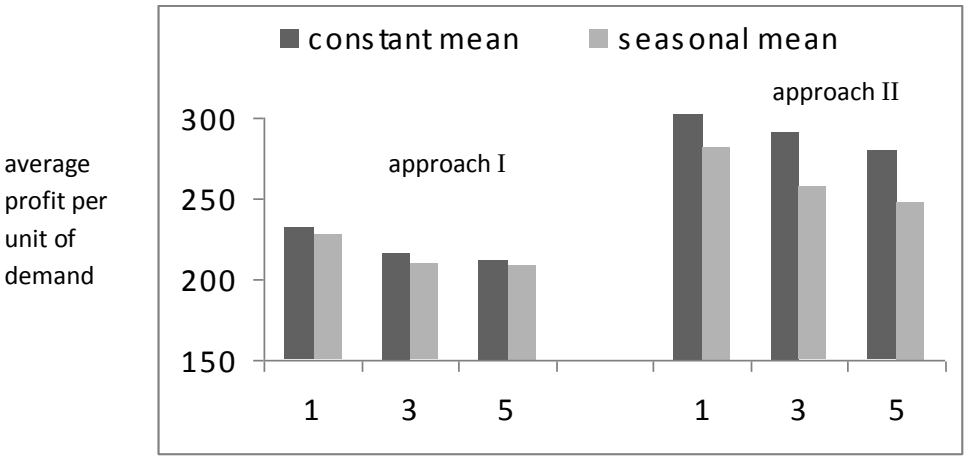
The rolling horizon approach performs 12-20% better than LP without update. However, it is seen that the performance gap reduces with increasing level of uncertainty which highlights the fact that rolling horizon cannot handle very high deviations from the expected values of uncertainty. This necessitates the use of solution methods that can explicitly account for uncertainty during optimization.


average inventory
300
200
100
0
constant
fluctuating
seasonal
avg product inventory
avg resource inventory

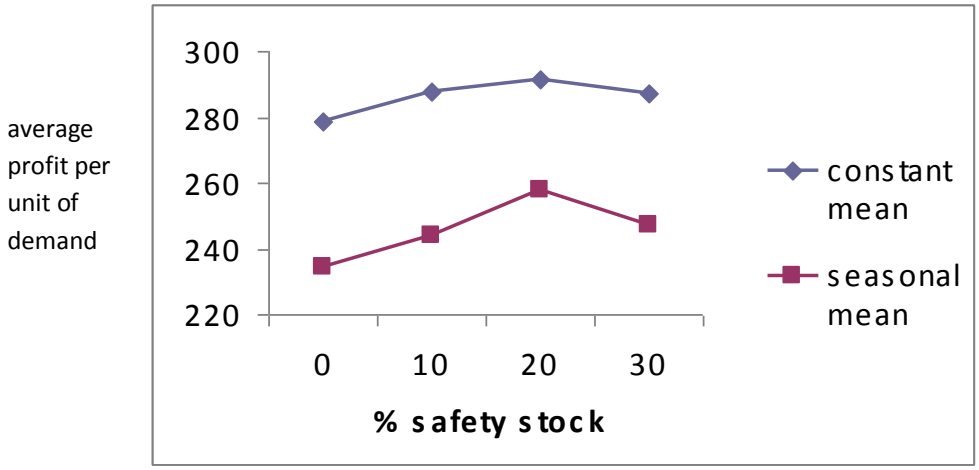
(a)


average profit per unit of demand
constant mean
seasonal mean
approach I
approach II
300
250
200
150
L
M
H
L
M
H

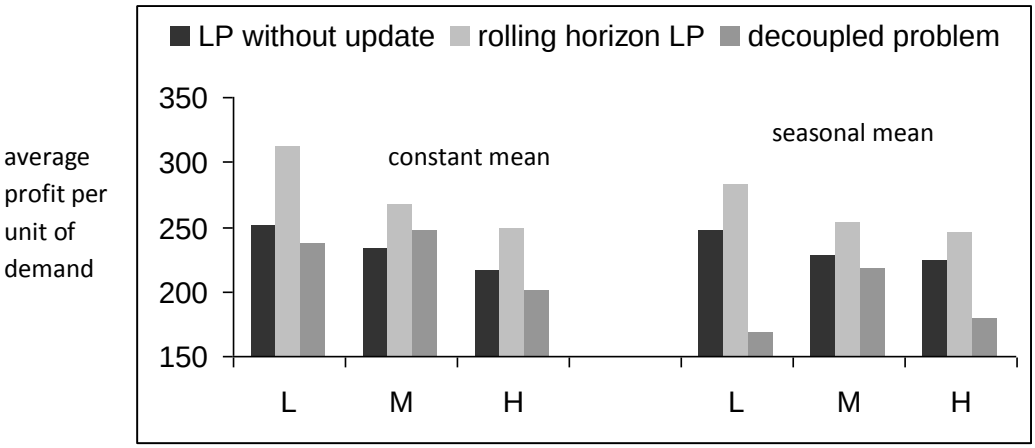
(b)


average profit per unit of demand
constant mean
seasonal mean
approach I
approach II
300
250
200
150
1
3
5
1
3
5

(c)



(d)



(e)

Figure 3.11: Results and comparison (a) Average inventories of products and resources for the deterministic problem. (b) Profit per unit of demand for each level of uncertainty (L-low, M-medium, H-high). The first set (3 bars) corresponds to solution approach I and the second set corresponds to solution approach II. (c) Profit per unit of demand for each value of lead time in resource order arrival (1,3 and 5 units). First set for approach I and second for approach II. (d) Profit per unit of demand v/s buffer stock as obtained by using solution approach II. (e) Profit per unit of demand v/s level of uncertainty. First set for demand with a constant mean and the second set for demand with a seasonal mean.

3.7. Conclusions

Mathematical programming models are developed to determine optimal resource procurement and allocation policies in conjunction with production planning, when the resource is perishable. A hierarchical approach is used due to the difference in time scales of planning and scheduling problems. Uncertainty in demand and resource life is addressed by a rolling horizon solution approach. It is shown that a significant loss in profitability is possible when the resource management is performed independent of production planning. The performance gap depends on the level to which the resource breaking rate is affected by usage, average resource life and variability in the total number of resources used per period. Stochastic optimization methods are required for improved decision-making by explicitly accounting for uncertainty during optimization. This is considered in the next chapter.

CHAPTER 4

AN APPROXIMATE DYNAMIC PROGRAMMING APPROACH TO SOLVING PLANNING AND SCHEDULING PROBLEMS WITH PERISHABLE RESOURCES

In the previous chapter, mathematical programs were developed for planning and scheduling with perishable resources. In the presence of uncertainty in product demand and resource life, the hierarchical framework was implemented in a rolling horizon fashion. This enabled the presence of a feedback loop to update the knowledge of system attributes at each time step. While solving the LP in a rolling horizon fashion, the most probable realization of uncertainty is considered. Therefore, the performance of the rolling horizon approach is determined greatly by the form of uncertainty. In general, the rolling horizon approach is well suited for problems where there is little likelihood of sudden and radical changes in the uncertain parameter. This is because the rolling horizon approach provides a reactive mechanism, which takes corrective action once the deviation from the most probable realization (or trajectory) is registered. In other words, the rolling horizon approach only has a myopic view of the uncertainty. For most applications however, the myopic view of the uncertainty does not suffice and proactive mechanism is desired. For such problems, exclusively accounting for the uncertainty during optimization is imperative. A particular example of problems requiring fore-sight with respect to parametric uncertainty is, when the problem attributes are changed permanently and drastically by decisions. In an inventory control problem, this situation may arise when the product demand inflates or falls rapidly, and the firm's performance during these scenarios determines its long run market share. The same is true for pricing problems. This aspect is explained by means of an illustration in 4.1. The perishable resource problem is then reconsidered and the planning problem is formulated as a Markov Decision Process (MDP) in section 4.2.1. MDP is an elegant framework to solve stochastic decision problems. Owing to the large state and action spaces, an approximate dynamic programming (ADP) algorithm is employed and discussed in detail in section

4.3. Since the uncertainty space is large as well, approximation of the expectation is performed during value function updates. MDP solutions are presented for three cases: (i) deterministic case, (ii) the case with an instance of the uncertainty model considered in Chapter 3, (iii) the case with product demands prone to inflation. The performance of the ADP algorithm is compared with that of the rolling horizon approach outlined in Chapter 3.

4.1 Inventory planning under inflationary or recessionary demand scenarios – an illustration

The *inventory control problem* is the one faced by a firm that must decide how much to order/produce in each time period to meet demand for its products. The ordering decision is made in the presence of demand uncertainty, so that there is a possibility of shortage or overage, pertaining to high and low realizations of the uncertainty respectively. The backorders and (or) extra stock can be carried over to the next time period. Typically, both shortage and overage are penalized, since the former leads to loss in potential revenues and the latter adds to the inventory holding cost. The optimal order quantity at each time would thus minimize the total cost including that of shortage and overage over the time horizon. This is illustrated by a simple example below:

Example I:

For simplicity, let us consider an infinite horizon problem with a single product and the future costs discounted by a factor of γ. $y1_t$ is the order quantity at time t. y_t is the on hand inventory at the end of time t and it is assumed that backorders are not allowed. The order arrives at the beginning of time t and the demand is realized at the end of the time epoch t. At each time, the demand for product, $\tilde{D}_t$, is either 4, 5 or 6 units, with equal probability i.e., 0.33. The cost function is shown in (4.1), where C_P and C_h are the costs of missing demand and holding inventory respectively. Cost associated with placing an order is also accounted for by considering the fixed ordering cost C_f and variable cost, C_v per unit of ordered quantity. Unlimited supply of product is assumed at each time. The inventory control problem presented above can be formulated as an MDP where the state comprises y_t and decision is $y1_t$. Since the demand scenarios are equi-probable at each time, the demand state does not need to be the part of the state description. The MDP is

solved using value iteration (described in section 2.4 of Chapter 2) and the optimal policy π_I is as shown in (4.2) for C_P =1 C_f= 0.3 and C_v=0.2, and C_h =0.1. It is a well known result (Hopp and Spearman, 1996), that the optimal policy is of the parametric form (*s*,*S*) when the uncertainty has an underlying Markov chain. The parametric form of the policy translates into placing an order whenever the inventory falls below *s* to bring it up to *S*.

$$E(y_{1t}, \tilde{D}_t) = C_h \max\{y_{t-1} + y1_t - \tilde{D}_t, 0\} + C_P \max\{\tilde{D}_t - y_{t-1} - y1_t, 0\} + C_f \delta(y1_t > 0) + C_v y1_t \quad (4.1)$$

$$y1_t = \begin{cases} 0 & \text{if } y_t > 4 \\ 5 - y_t & \text{otherwise} \end{cases} \quad (4.2)$$

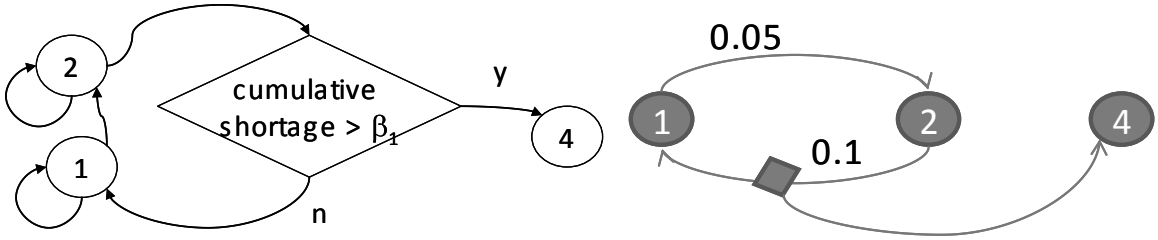
A variation in the demand model is considered next. Example II is used to illustrate the dependence of future demands on firm's performance during a temporary surge in demand. The example is motivated by the fact that if the firm continually misses demand, the goodwill of the firm would be affected. This might result in permanent loss of some customers or loss of market share. Consequently, the firm may see reduced demand when the inflation in demand subsides.

Example II:

Figure 4.1(a) shows a demand model where the product demand can be in one of the three states $\zeta \in \{1,2,4\}$. The first state, i.e., ζ=1 is similar to one in example I, where demand can be 4,5 or 6 units with equal probability (designated as 5±1). This is referred to as the *nominal demand* state. At each time, with finite probability, demand may transition to an inflationary state designated by circle 2. State 2, i.e., ζ=2, represents the case of increased demands for goods due to boom in the economy, industry growth or other socio-economic or technology related factor. Once in state 2, the system may transition back to nominal demand state, i.e., ζ=1 with finite probability at each time. The demand in state ζ=2 is 10±1, where all three values are equi-probable. However, if the overall shortage during the inflationary state exceeds a certain limit β_1, the firm permanently loses market share and the demand transitions to state ζ =4, which is lower

than the nominal demand. This is referred to as the *reduced-nominal* demand state. The demand in this state is 4±1. It is required that the order quantities not exceed 10 units at each time. The MDP formulation of the above problem is an extension to example I. The system state must now contain more information in addition to on-hand inventory, like the current demand state and cumulative shortage when in demand state 2. The demand model is represented by Figure 4.1(a) and the transition probabilities are shown as arrows of the equivalent Markov chain in Figure 4.1(b). The diamond in Figure 4.1(b) represents the test of whether the cumulative shortage during demand state ζ =2, exceeded the predefined limit β_1, and the circles represent demand states. Decision has to be made for order quantities at each time so as to maximize profit (4.3) in this case. This is accomplished by adding the revenue from product sales at C_P per unit. Consequently the shortage cost is eliminated from the expression in (4.1). It is assumed that each product sale brings revenue of C_P units. The ordering cost has fixed and variable components (C_f and C_v) similar to example 1, and unit holding cost is given by C_h.

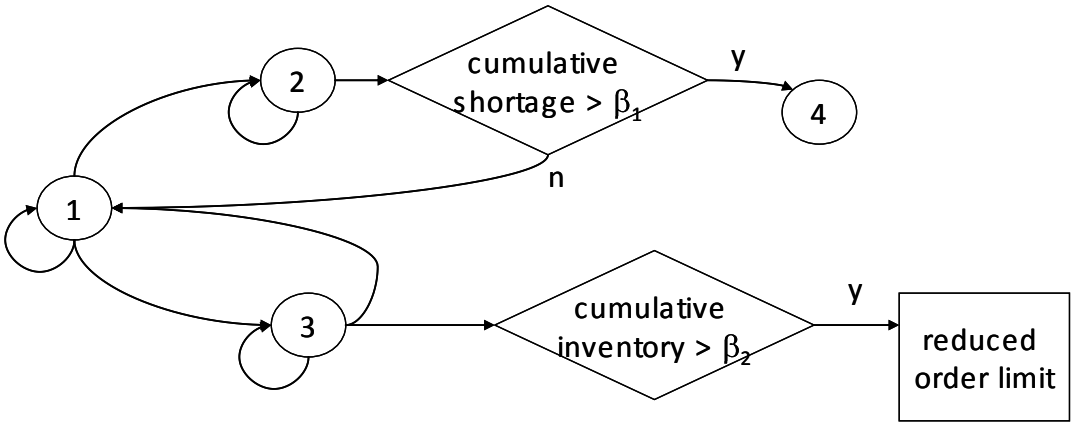
$$P(y_{1t}, \tilde{D}_t) = C_P \tilde{D}_t - C_P \max\{\tilde{D}_t - y_{t-1} - y1_t, 0\} - C_h \max\{y_{t-1} + y1_t - \tilde{D}_t, 0\} - C_f \delta(y1_t > 0) - C_v y1_t \quad (4.3)$$



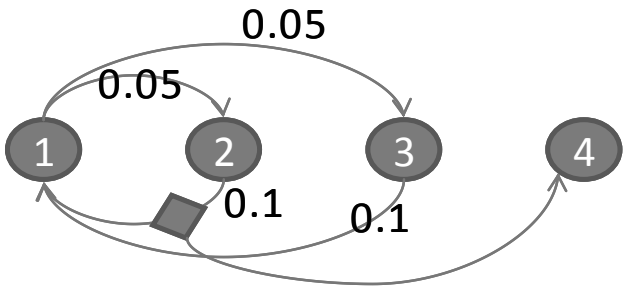
(a) (b)

Figure 4.1: (a).A demand model involving inflationary demand scenario. (b) Illustration of transition probabilities of the Markov chain

The parameter values are shown in Table 4.1(a) and the optimal policy (π_{II}) for the considered parameter values is reported in Table 4.1(b). It turns out that the optimal policy is still of the parametric form (*s*,*S*) and depends on the on-hand inventory and demand realization but is independent of the cumulative shortage. The optimal policies pertaining to each individual demand state ($\pi_I^{\zeta}, \zeta = 1,2,4$) are also reported. The two policies are different for ζ =1,2,4, π_I^{ζ} being the sub-optimal policy for the problem in example II. $\pi_I^{\zeta}, \zeta = 1,2,4$ are obtained by using demand states ζ =1,2 and 4, individually in example I. It must be noted that $\pi_I^{i}, i = 1,2,4$ would be the best possible policy obtained if the problem in example II was solved using a rolling horizon approach.



(a)



(b)

Figure 4.2: (a) A demand model involving inflationary and recessionary demand scenarios.(b) Illustration of transition probabilities of the Markov chain

This is because the algorithm would fail to consider the possibility of demand transition owing to low probability values. On the other hand, policy π_{II}^{ζ} for ζ=1 is more aggressive and orders more in view of the upcoming demand inflation, while the reverse is true for ζ=2. Since ζ=3 is the absorbing state, the corresponding two policies are understandably the same.

Example III:

An additional demand state is considered in this example as shown in Figure 4.2(a). From the nominal state, demand may also transition to a recessionary state 3. This state is marked by a steep decline in product demand due to economic recession, defamation of the product, introduction of a better substitute etc. and demand is likely to be restored to nominal state in future. Since the demand falls suddenly, a lot of inventory may be accumulated if the firm does not plan ahead. If the cumulative inventory during recessionary demand exceeds β_2, the firm is required to renegotiate supplier contracts to permanently reduce the limit on order quantities. This is called *reduced-order* limit. The downside is that, even if the demand is restored to normal, the firm may not order enough goods and thereby lose demand continuously. The transition probabilities are shown in Figure 4.2(b). Together with the attributes listed in example II, the system state contains cumulative inventory during recessionary demand and current order limit. The optimal policy (π_{III}) obtained by solving the MDP is reported in Table 4.2(b) for the parameter values shown in Table 4.1(a). It is seen that the parametric (*s*,*S*) form of the policy is still maintained and the policies corresponding to the individual demand states $\pi_I^i \ i = 1,2,3,4$ are more similar to π_{III} as compared to π_{II}. This is attributed to the fact that in example III, there is a possibility of an increase as well as decrease in product demand. The two factors, therefore, compensate for one another.

Table 4.1: Parameters and policies for the inflationary demand scenario (a) Parameter values for the inventory control problem, (b) Policies corresponding to approaches I, II

Cp	Cf	Cv	Ch
1	**0.3**	**0.2**	**0.1**

(a)

policy		1	2	4
π_{II}	**s**	4	9	3
	S	6	10	4
$\pi_I^{\zeta}, \zeta = 1,2,4$		3	10	3
		4	11	4

(b)

Table 4.2: Policies corresponding to approaches I and II, for the inflationary and recessionary demand scenario

policy		1	2	3	4
π_{III}	**s**	4	9	2	4
	S	6	11	4	5
$\pi_I^{\zeta}, \zeta = 1,2,3,4$		5	10	3	4
		6	11	4	5

From the above exercise, it may be concluded that for a general system where the complete structure of uncertainty is not revealed in a short horizon, ADP methodology has a high potential for improved performance as compared with a rolling horizon approach.

For the PR problem, the planning level bears more far reaching effects and therefore calls for rigorous treatment of uncertainty. The formulation of the planning problem as a MDP is presented in the next section. The reader is referred to Chapter 3 for problem details and assumptions.

4.2 Formulation of the perishable resource problem as MDP

To be able to successfully formulate the PR problem as MDP, the most important requirement is the Markov property or the memory-less property. This implies that, the next state and stage-wise reward must depend only on the current state and action. The resource life is independent of past decisions and states. Also, since the demand for products was assumed to have an underlying Markov chain in Chapter 3 and in examples presented in section 4.1, this condition is met. Finally, for simplicity of illustration, we opt to formulate the infinite horizon problem which pertains to a stationary optimal policy. A discount factor γ=0.9 is used in all cases. Demand model is represented by $d(\zeta)$ for general discussion where ζ=1,2,..n_D, designates the states of the Markov chain and $d(\zeta)$ the demand associated with that state. Specific demand models are deferred until section 4.3.3.

4.2.1 MDP formulation

As discussed in Chapter 2, a MDP is characterized by the tuple (S,A,T,R,γ) whose descriptions in the context of the PR problem are shown below. For consistency, notational convention is followed from Chapter 3 as much as possible.

- *State*

  The state at time t, (s_t) includes four pieces of information:

  (i) Product/stone inventory -x_{it} $\quad i=1,2,\ldots n_P$

  (ii) Resource/mold inventory – y_{ijt} $\quad i=1,2,\ldots n_P;\ j=1,2,\ldots\ \theta_T$

  (iii) Product demand scenario - ζ_t, $\quad \zeta_t \in \{1,2,..n_D\}$

  (iv) Past orders - $y1_{i,t1}$ $\quad i=1,\ t1=t\text{-}1,\ t\text{-}2,\ldots t\text{-}l$

$$s_t = [x_{it}\ \ y_{ijt}\ \ k_t\ \ y1_{i,t1}] \tag{4.4}$$

$$dim(s_t) = n_P + n_P * \theta_T^{\max} + 1 + l \quad (4.5)$$

$$|S| = (x_{\max})^{n_P} \times (y_{\max})^{n_P \theta_T^{\max}} \times n_D \times (y1_{\max})^{l} \quad (4.6)$$

The state description, state dimension and the size of the state space is given by (4.4), (4.5) and (4.6) respectively. $x_{\max}$ and $y_{\max}$ are externally imposed limits to keep the problem size finite and ensure that an unreasonable size inventory is not held in the system. $y1_{\max}$ is the upper limit on reorder quantities of the resource and $\theta_T^{\max}$ is the maximum possible resource life.

- *Action*

The actions/decisions are clearly defined in Chapter 3 and classified as planning and scheduling level decisions. The three types of actions at time *t* are:

(i) Production quantities - $x1_{it}$ $i=1,2,\ldots n_P$

(ii) Resource reorder - $y1_{it}$ $i=1$

(iii)Resource transition and allocation - $y2^{sch}_{i,i1,j,t}$ $i=1,2...n_P$; $i1=1,2...n_P$; ; $j=1,2,.\ \theta_T$,

$y3^{sch}_{i,j,j1,t}$ $i=1,2...n_P$; $j=1,2,.\ \theta_T$,$j1=1,2,..,j-1$

As outlined in Chapter 3, the production and resource ordering decisions are planning level decisions while the resource transition and allocation decisions are determined at the scheduling level in order to best meet the production requirements. (The time scales *t* and *t'* for the two levels are shown again in Figure 4.3). Therefore, only the production and resource ordering decisions are considered for the planning problem. The resource allocation decisions are obtained by solving the scheduling problem (thereby indicated by suffix *sch*), a discussion on which is deferred until section 4.3.3. Since only the light molds are ordered at any time, the resource reorder decision contributes only a single dimension to the decision space.

$$a_t = [x1_{it} \ \ y1_{it}] \quad (4.7)$$

$$dim(a_t) = n_P + 1 \quad (4.8)$$

$$|A| = x1_{\max}^{n_P} \times y1_{\max} \tag{4.9}$$

The description of action, action dimension and the size of the action space is given by (4.7), (4.8) and (4.9) respectively. Again $x1_{\max}$ and $y1_{\max}$ are limits on production and reorder quantities respectively.

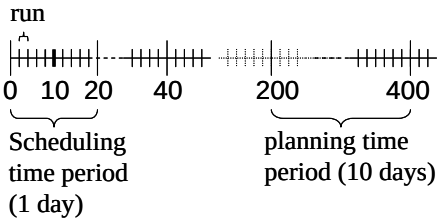


Figure 4.3: Time scales of the planning and scheduling problems

- *State transition function*

The probabilities of state transition are dependent on two sources of uncertainty:

(i) Product demand -$\tilde{D}_i \in d_i(\zeta)$

(ii) Resource life distribution -$\tilde{A}_j \in \aleph(\mu, \sigma)$

While product demand affects the product inventory, resource life distribution determines the resource inventory at the next time step. (4.10) and (4.11) show the transition equations for product and resource inventory at the next time step. Depending on the combined realization ω of the uncertain parameters $\tilde{D}_i$ and $\tilde{A}_j$, the state at the next time is obtained.

$$x_{i,t+1}^{\omega} = \max(x_{it} + x1_{it} - \tilde{D}_{it}, 0) \tag{4.10}$$

$$y_{ij,t+1}^{\omega} = \sum_{\mathrm{i1}} \mathrm{y2}_{\mathrm{i1,i,j,t}}^{sch} + \sum_{j1} y3_{i,j1,j,t}^{sch} - \sum_{i1} y2_{i,i1,j,t}^{sch} - \sum_{j1} y3_{i,j,j1,t}^{sch} + y1_{i,t-l}\,\tilde{A}_j \tag{4.11}$$

- *Stage-wise reward*

The reward function (4.12) consists of the following terms:

(i) Revenue from product sales

(ii) Cost of procuring new resources

(iii) Cost of production

(iv) Cost of holding product and resource inventory

$$r(s_t, a_t) = \sum_i R_i (x_{it} - \max(x_{it} - \tilde{D}_{it}, 0)) - \sum_i P_i x1_{it} - \sum_i \sum_j E_i y1_{ijt} - \sum_i H_i x_{it} - \sum_j \sum_j H_{ij} y_{ijt} \quad (4.12)$$

The objective is to determine a policy π that maximizes the infinite horizon discounted reward $V^{\pi}(s)$ when the initial state is s (4.13).

$$V^{\pi}(s) = \max_{\pi} \mathrm{E}(\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \mid s_0 = s) \quad (4.13)$$

In order to determine the state at the next time step, resource transition and allocation decisions need to be taken. They are determined at the scheduling level. Due to the absence of sources of uncertainty at the scheduling level, a linear program (LP) similar to M2 in Chapter 3 may be solved. However, in order to achieve computational efficiency at the planning level, a heuristic is developed to determine the resource transition and allocation decisions. The heuristic is guided by the solution of LP described by model M2 in Chapter 3.

4.2.2 A heuristic to obtain resource transition and allocation decisions

The resources are allocated in a way that prioritizes the fulfillment of production plan for the planning cycle. This is done through the path of minimum color grade transitions in order to ensure maximum possible flexibility in the future. The heuristics are schematically shown in Figures 4.4(a) and 4.4(b). The details are presented below:

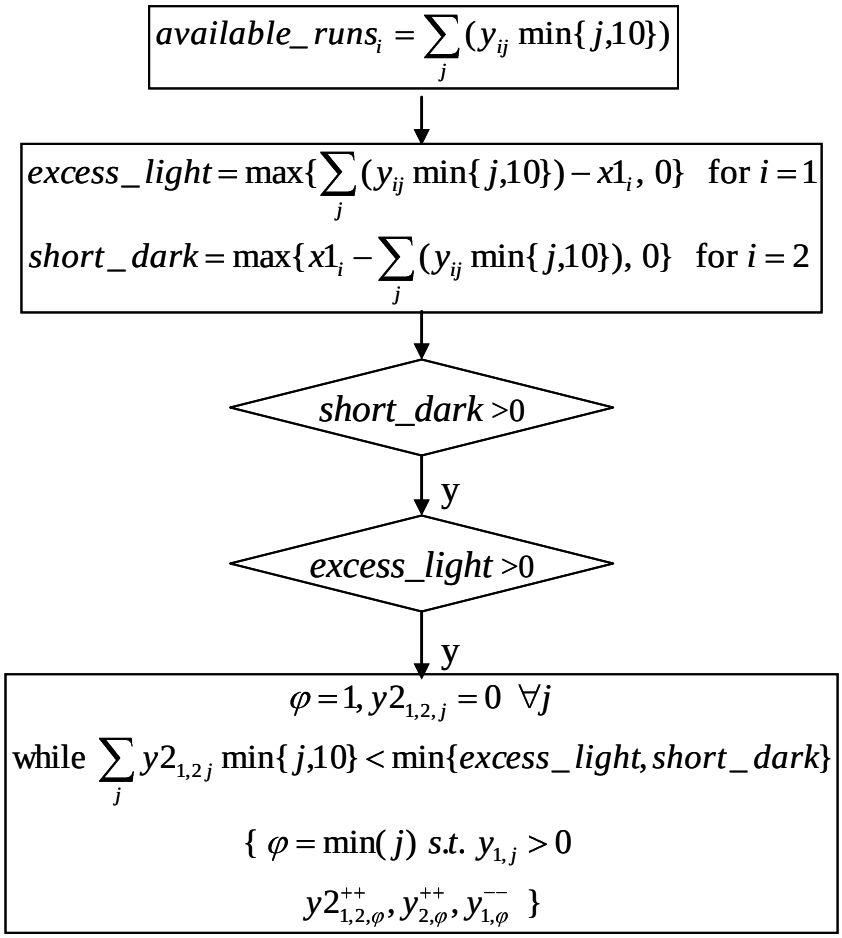
The resource transition heuristic

Since it is assumed that only light resource (i=1) can be ordered, resource transition needs to be performed for production of dark products (i=2). In order to determine the number of possible transitions, excess light runs, *excess_light* are determined at the beginning of the scheduling horizon. Since a resource can be used a maximum of 10 times (scheduling horizon is 10 days and the resource can be used only once a day), the number of available runs needs to be adjusted. The adjustment needs to be made for resources with more than 10 runs remaining. Now, the shortage in dark resources is determined in terms of number of runs, *short_dark*. The smaller of the two quantities, i.e. excess_light and short_dark is scheduled for transition. While transitioning, the oldest resources are prioritized. This ensures that maximum resource inventory is held as light resources, imparting flexibility in the future.

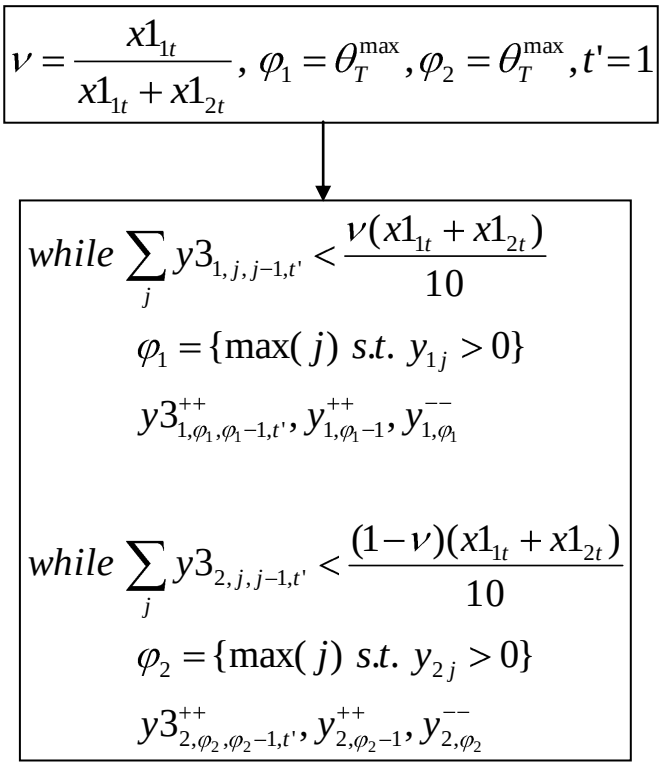
The resource usage heuristic

To be able to successfully allocate the resource to production, production quantities need to be determined at each scheduling time period. The total production requirement in the given planning period, i.e. $x1_i$, i=1,2 is equally divided into 10 days. Moreover, the overall fraction of light products, i.e. v, is maintained during each scheduling time epoch t'. Having determined the production quantities, resources are allocated based on *newest-first* rule. This is because when the resource life distribution is very uneven, the resources with more number of runs left must be utilized early so as not to run out of resources.

Having obtained a simple heuristic to determine the state transitions, we analyze the size of the MDP problem and present an ADP algorithm for its solution in the next section.


$available\_runs_i = \sum_j (y_{ij} \min\{j,10\})$
$excess\_light = \max\{\sum_j (y_{ij} \min\{j,10\}) - x1_i, 0\}$ for $i = 1$
$short\_dark = \max\{x1_i - \sum_j (y_{ij} \min\{j,10\}), 0\}$ for $i = 2$
$short\_dark > 0$
y
$excess\_light > 0$
y
$\varphi = 1, y2_{1,2,j} = 0 \ \forall j$
while $\sum_j y2_{1,2j} \min\{j,10\} < \min\{excess\_light, short\_dark\}$
$\{ \varphi = \min(j) \ s.t. \ y_{1,j} > 0$
$y2^{++}_{1,2,\varphi}, y^{++}_{2,\varphi}, y^{--}_{1,\varphi} \ \}$

(a)



(b)

Figure 4.4. Heuristics for (a) resource transition and (b) resource usage at scheduling level

4.3. Solution of the MDP

4.3.1 Problem size

The state space

Using the expressions in (4.5) and (4.6) and parameter values shown in Table 4.3, the state dimension and size of the state space may be obtained as below:

$dim(s)$ - 2 + 40 + 1 + 3 = 46

$|S|$ - $400^2 \times 100^{40} \times 12 \times 100^3 \sim 1.9*10^{93}$

It must be noted that the term $(y_{\max})^{n_P \theta_T^{\max}}$ contributes the most to problem size. While implementing the ADP algorithm, we will use an aggregation scheme to effectively reduce the problem size on this account.

The action space

For the parameter values shown in Table 4.3, the dimension of the action/decision variable and the size of the action space is as shown below:

$dim(a)$ - 2+1 = 3

$|A|$ - $200^2 \times 20 = 8 \times 10^5$

The assumption of maximum capacity utilization is made to further reduce the action space. E.g., if the plant capacity is 200 per planning cycle and $x1_{it}$=160 for i=1, then $x1_{it}$ =40 for i=2, given that the resource requirement is met. Therefore, the dimension of a_t is now 2 and $|A| = 200 \times 20 = 4000$.

The uncertainty space

Although the demand scenarios are finite, the life of each incoming resource is drawn from a normal distribution. With $\theta_T^{\max}$ =20, there are practically infinite possibilities for the resource life distribution at each time even after coarse approximation of the normal distribution. Consequently, the computation of expectation in the Bellman equation presents a computational challenge.

Since the state and uncertainty space are very large, exact solution methods may not be used to solve the MDP. In order to circumvent the difficulty posed by large problem size, an approximate dynamic programming algorithm is employed. The algorithm is presented in the following section.

Table 4.3: Parameter values for the perishable resource problem

Parameter	n_p	$\theta_T^{\max}$	n_D	x_{max}	y_{max}	$x1_{max}$	$y1_{max}$	l
value	2	20	12	400	100	200	20	3

4.3.2. An Approximate Dynamic Programming algorithm

As noted in section 2.2, MDP*s* are traditionally solved using dynamic programming and optimal solution is obtained by solving the well known Bellman equation (4.14). The optimal policy is obtained as a function of system state (4.15) (Puterman 1994) presented in section 2.2 of Chapter 2.

$$V(s) = \max_{a \in A_s} \{ r(s,a) + \sum_{s' \in SS} p(s' \mid s,a) V(s') \} \quad \forall s \tag{4.14}$$

$$a^*(s) = \arg\max_{a \in A_s} \{ r(s,a) + \sum_{s' \in SS} p(s' \mid s,a) V(s') \} \quad \forall s \tag{4.15}$$

However, when the state space is large, it becomes practically impossible to solve (4.14) for each states $s \in S$. Additionally, when the state transition matrix is not sparse (high uncertainty), the exact computation of expectation is not viable. Therefore, we make use of approximate dynamic programming (Bertsekas 1995) also called real time dynamic programming (RTDP) (Sutton and Barto, 1998). The central idea of the algorithm is that, instead of working with the entire state space, a subset of states or *sample space* is maintained as prototype states. Moreover, the set of prototype states or the reduced state space evolves over time as we step forward in time. Since the entire state space is not considered, the value function needs to be approximated for the states s' when they are not part of the *sample space*. To this end, a local approximation in the form of *k-nearest neighbors* (Hauskretch 2000) is used. The method draws on the notion that similar states would exhibit similar behavior in terms of associated value function and decision-rule. The parity among the states is described by a Euclidian distance measure. As the distance increases, the states become increasingly different. The same notion of parity between the states is exploited to keep the size of the sample space from getting very large. Each state in the sample space is seen as representing a cluster of neighboring states, lying within a circle around it. The circle can be drawn using the same Euclidian distance as criterion for radius.

The algorithm is schematically shown in Figure 4.5. The sample space designated as *S*, is initialized with the pre-specified starting state s_0. If not specified, a random state can be chosen as s_0. The iteration counter, set as n=1 represents one complete experiment

until the end of horizon *H*. *t* is the time counter which goes from 0 to *H* and resets to 0 for each new iteration. The value function for the s_0 is initialized and an iteration given by (4.16) is performed. During the value iteration, the value function for the next state s' is determined by using an aggregation scheme *compress* and *k-nearest neighbor* interpolation as described below:

State aggregation function *compress*: Since the dimension of the state is high and the resource inventory is the single-most contributing factor, the state is aggregated for the purposes of maintaining the sample space and value function determination. The aggregation scheme lumps together the resources with several lives into two large buckets *new* and *old*. If the resource has more than 10 runs remaining, it is termed as new, otherwise old. A finer aggregation is also possible. Therefore, the dimension of the state is greatly reduced in leu of lost information. As an additional data, the total number of runs available from all the resources is also made a part of the state description. The aggregation is represented by *compress* and the aggregated state is identified as s^c as shown in (4.17).

$$s^c = compress(s) = [x_i \ \sum_{j=1}^{10} y_{ij} \ \sum_{j=11}^{20} y_{ij} \ \sum_{j=1}^{20} jy_{ij} \ y1_{t1} \ \zeta] \quad for\ i = 1,2,n_P, t1 = 1,2,..l \qquad (4.17)$$

Value function approximation – The Euclidean distance δ_s (subscript *s* for distance from state *s*) used to establish similarity between a query state $s_Q^{'c}$ and a state $s^c \in S$ is shown in (4.18). In (4.18), subscript $m \in \{1,2,..dim(s)\}$ denotes the dimensions of the state and w_m are parameters that are used to place weights on different dimensions. It is believed that a difference in the values of certain dimensions of the state would cause the value function to change greatly across different states as compared with others. For such dimensions, a large weighting factor w_m is suitable. For notational clarity, the subscript *c* for the aggregated states is suppressed. Once the distance between the query state and all the states (that belong to the sample space) is known, the top *k* states sorted by ascending order of the distance δ_s, are chosen for interpolation. The *k* states are generally referred to as *nearest* neighbors. The nearest neighboring states are designated as $s=\{1,2,\ldots k\}$ with some abuse of notation. The value function is now computed by (4.19). If $|S| < k$, $|S|$

neighbors are used. Since $\frac{1}{\delta_s} \geq 0$ and $\sum_{s=1}^{k} \frac{1}{\delta_s} = 1$, the interpolation is convex and the contraction map is preserved.

$$\delta_s = \left(\sum_{m=1}^{\dim(s)} w_m \{ s_Q^{'}(m) - s(m) \}^2 \right)^{1/2} \tag{4.18}$$

$$V^{kNN}(s_Q^{'}) = \sum_{s=1}^{k} \frac{1}{\delta_s} V(s) \tag{4.19}$$

While attempting to perform the value iteration, the maximization is performed over all actions in the action space. Since *dim*(*a*) is small, a homogenous grid is used to reduce the size of the action space. E.g., resource order can only occur in multiples of 4 and production is assumed to take place in batches of 10. Additionally, for a state-action pair, many next states are possible owing to large uncertainty. Since exact computation of expectation over all possible next states s′ is not possible, sampling methods are used as described below.

Approximation of expectation: The idea of approximation of expectation is simple. Starting from s_0, for each action $a \in A$, samples of next states $s'=f(s_0,a,\omega)$, $\omega \in \Omega$ are obtained, where Ω is a set of sampled values of uncertain parameters. $\omega \in \Omega$ also have probabilities associated with them. These are designated as p_ω and are re-normalized to ensure convergence. The value iteration update based on sampled uncertainty is shown in (4.20)

$$\hat{V}_{n+1}(s) = \max_{a \in A_s} \{ r(s,a) + \frac{\sum_{\omega \in \Omega} p(\omega \mid s,a) V_n(f(s,a,\omega)}{\sum_{\omega \in \Omega} p(\omega \mid s,a)} \} \tag{4.20}$$

In Figure 4.5, the approximation is illustrated for only one sample realization of the uncertainty. Higher number of samples helps convergence.

After the value iteration update, one or a set of maximizing actions a^* are obtained. a^* is the best action for state s_0 based on the knowledge so far. However, because of the approximation of expectation mentioned above, the value function is a biased estimate of the mean discounted reward. Also, since all the states in the sample space are not updated during one iteration, exploration needs to be performed to access different parts of the original state space. This is accomplished by perturbing the action around the best known action a^*. This is described below.

Exploration v/s exploitation: The perturbation in the action is performed by using $a^*+\Delta a^{\{n\}}$ instead of a^* to move forward in time. The perturbation $\Delta a^{\{n\}}$ is dependent on the iteration counter n. At the beginning of the algorithm, the value function estimates are poor and very few states are a part of the sample space for successful interpolation and good representation of the original state space. This is the reason why higher degree of perturbation in action is needed. As the number of iterations increase, Δa is dampened to be able to exploit the information obtained so far. The specific form of Δa used for the PR problem is shown in (4.21). q represents a general quantity, e.g., production and reorder, $\lambda_1 \in (0,1]$, $\lambda_2 > 0$ are parameters and q_{max} is the maximum possible value of q. For the perishable resource application $\lambda_1=0.4$ and $\lambda_2=1.5$ are used.

$$\Delta q = q^* \pm (\lambda_1 q_{\max})^{\lambda_2 n} \tag{4.21}$$

Having performed the value iteration update for the initial state s_0, and obtained the perturbed action $a^*+\Delta a^{\{n\}}$, next state s_1 is obtained by taking one sample realization of the uncertainty ω_1. In usual practice a sample trajectory for the experiment, i.e., $\omega_1,\omega_2,..\omega_H$ is generated a priori. At this point, it needs to be determined whether s_1^c must be admitted to the sample space.

Conditional admittance to sample space - The next state s_1^c is not admitted if it has enough representation in the sample space, i.e. there is at least one state in the sample space, that is similar to s_1^c. The state is referred to as an *alias*. An alias is also determined based on the distance criterion seen in (4.18). A neighbor is called an alias if its distance

from the query point is below a pre-defined threshold δ^{max} (4.22). Having obtained s_1, the time counter is increased and the iteration is repeated. If s_1 was admitted to the sample space in the previous iteration, its value function would be assigned by the next value iteration. Otherwise, the aliases are used as s_1 for updating the value function. A small value of δ^{max} would result in a very large state space whereas a large value of δ^{max} would result in poor representation of the relevant state space by the sample space. Therefore, δ^{max} is an important parameter which must be chosen carefully.

$$alias(s_Q) = \{s : \delta(s_Q, s) \leq \delta^{\max}\} \tag{4.22}$$

As pointed out earlier, the value function estimate obtained after value iteration is a biased of the true mean, a smoothing parameter α_η is used.

Value function smoothing - This technique essentially determines a moving average estimate of the value function $V(s^c)$ $s^c \in S$. Represented by (4.23), the value function update at the n^{th} iteration is obtained as a convex combination of the old and new values. The smoothing factors $0 \leq \alpha_\eta \leq 1$ are generally different for each state in that they depend on the number of times the corresponding state has been visited (designated by $\eta^{\{s\}}$). Therefore $\eta^{\{s\}}$ needs to be stored for each state $s^c \in S$. In order to strike the balance between exploration and exploitation, much the same way as with the action, α_η must be dampened with time as well. Expression (4.24) is used to achieve the same with λ_3=20 for the perishable resource problem.

$$\alpha_\eta = \frac{\lambda_3}{(\lambda_3 + \eta)} \tag{4.24}$$

When the incumbent state does not belong to the state space, the aliases are used for value function update. The iteration is repeated until either the computational limit is reached or the value function convergence criterion is met.

Step 0. Set n=1, define the starting state s_0,

$$s_0^c = compress(s_0), \text{ set } S = s_0^c, \text{initialize} V(s_0^c), \eta(s_0^c) = 1$$

Step 1. Set t=0, determine a sample path $\omega_1, \omega_2, .. , \omega_H$

Step 2. $\forall a \in A$, determine $s'_a = f(s_t, a, \omega_t)$

$$s'^c_a = compress(s'_a) \quad \forall a$$

Step 3a. $\forall a$, if $s'^c_a \in S$, determine $V(s'^c_a)$

else $V(s'^c_a) = KNN(V(s^c \in S))$

Step 3b. $$v(s_t^c) = \max_{a \in A} \{r(s_t, a) + \gamma V(s'^c_a)\}$$

$$a^* = \arg\max_{a \in A} \{r(s_t, a) + \gamma V(s'^c_a)\}$$

Step 4. If $s_t^c \in S$,

$$\overline{V}(s_t^c) = (1 - \alpha_\eta) V(s_t^c) + \alpha_\eta v(s_t^c) \qquad (4.23)$$

$$\eta(s_t^c) = \eta(s_t^c) + 1$$

elseif $alias(s_t^c) \neq \phi$

$$\overline{V}(s) = (1 - \alpha_\eta) V(s) + \alpha_\eta v(s_t^c) \quad \forall s \in alias(s_t^c)$$

$$\eta(s) = \eta(s) + 1 \quad \forall s \in alias(s_t^c)$$

else

$$add\ s_t^c\ to\ S,\ \overline{V}(s_t^c) = v(s_t^c),\ \eta(s_t^c) = 1$$

Step 5. If t<H,

$$a_t = a^* + \Delta a, \quad s_{t+1} = f(s_t, a_t, \omega_t),\ V = \overline{V}$$

set t=t+1 and go to *Step 6*

else go to *Step 1*

Step 6. If n<N, set n=n+1 and go to *Step 1*

else Stop

Figure 4.5: An approximate dynamic programming algorithm

The number of states as a function of iteration count n is shown in Figure 4.6(a) for a particular demand model (described in the next section). Selected states are added as they are encountered during each iteration. More states are added in the initial iterations and the number of new states to be added to the sample space dwindles as the sample space becomes a good representation of the relevant state space. A good strategy is to start with a high value of δ and keep reducing it until the performance for two successive values is comparable. Really high values of δ result in very few states in the sample space as seen in Figure 4.6(a). Also, the convergence of the value function for some of the most visited states is shown in Figure 4.6(b). Depending on the initialization, the value function either trends upwards or downwards and finally converges after sufficient number of iterations.

4.3.3. Demand modeling revisited

Three different demand models are considered:

(i) Deterministic demand- The seasonal demand pattern (4.25) used in Chapter 3 (section 3.4) is used. The demand pattern (4.25) in assumed to repeat until $t = \infty$, while counter τ=1,2,..30 is included in the state definition. The counter resets to 1 after τ=30.

$$d^{sea}(\tau) = \begin{cases} [60 \;\; 60] \;\; if \quad 1 \le \tau \le 10 \\ [140 \;\; 140] \;\; if \quad 11 \le \tau \le 20 \\ [100 \;\; 100] \;\; if \quad 21 \le \tau \le 30 \end{cases} \tag{4.25}$$

While implementing the ADP algorithm shown in Figure 4.5, for the deterministic problem, the smoothing operation (4.23) is not needed. This is because of the absence of uncertainty leading to exact value update for every state. The action is still perturbed to counter the effects of dicretization.

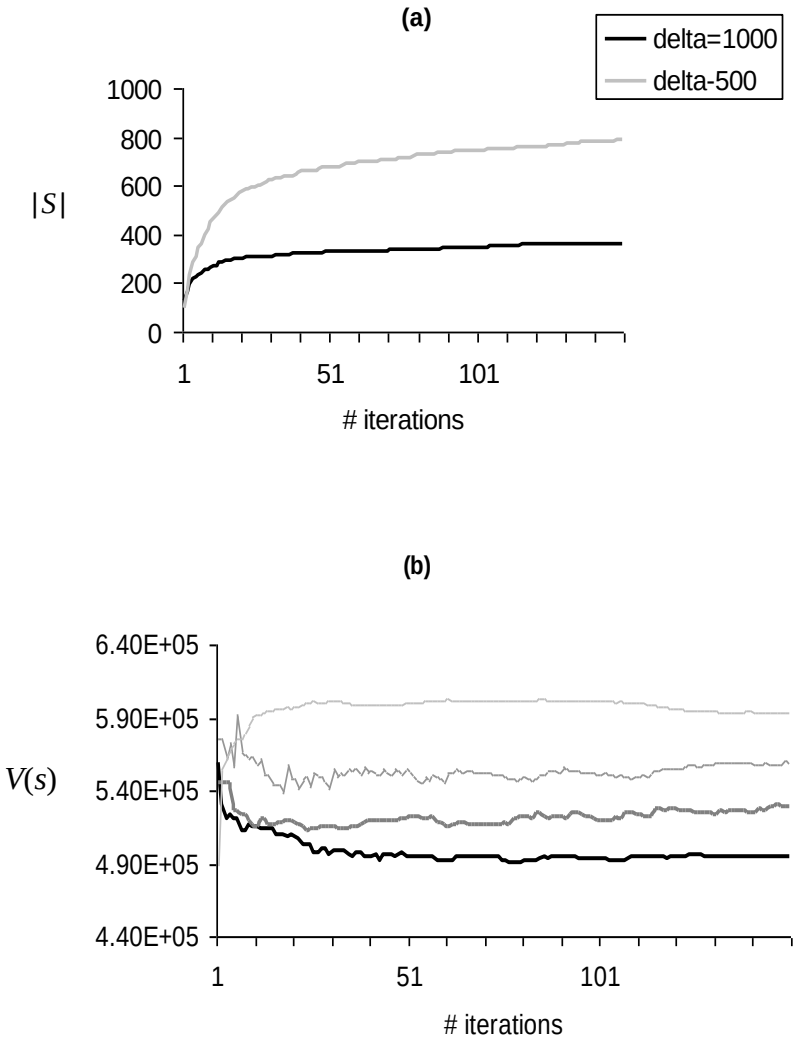


Figure 4.6: (a) number of states growing with iterations, (b) Convergence of value function

(ii) Demand with fluctuations around a constant mean

Similar to Chapter 3, the stochasticity in demand is modeled as a Markov chain with twelve states shown in Table 4.4. These states represent deviations from nominal demand for the light and dark products. The transition probabilities pertaining to low, medium and high cases of uncertainty are given by $T1$ (4.26), $T2$ (4.27) and $T3$ (4.28) respectively.

$$T1(\zeta,\zeta') = \begin{cases} 0.8 & if\ \ \zeta = \zeta' and\ \zeta' \le 4 \\ \dfrac{0.2}{3} & if\ \ \zeta \ne \zeta'\ and\ \ \zeta' \le 4 \\ 0 & otherwise \end{cases} \tag{4.26}$$

$$T2(\zeta,\zeta') = \begin{cases} 0.8 & if\ \ \zeta = \zeta' and\ \zeta' \le 8 \\ \dfrac{0.2}{7} & if\ \ \zeta \ne \zeta'\ and\ \ \zeta' \le 8 \\ 0 & otherwise \end{cases} \tag{4.27}$$

$$T3(\zeta,\zeta') = \begin{cases} 0.8 & if\ \ \zeta = \zeta' \\ \dfrac{0.2}{11} & otherwise \end{cases} \tag{4.28}$$

Table 4.4: Deviations from nominal demand for the light and dark products

s	L	D
1	0%	0%
2	-25%	10%
3	10%	-25%
4	20%	20%
5	-60%	40%
6	40%	-60%
7	-40%	60%
8	60%	-40%
9	45%	45%
10	-40%	-40%
11	-70%	-70%
12	65%	65%

The high uncertainty case corresponding to probability transition matrix $T3$ is considered around a constant mean of [100 100].

(iii) Inflationary demand case

In addition to the regular demand scenario in (ii) above, an inflationary demand scenario and a less-than-regular scenario is considered similar to example II. (To avoid confusion, the first demand scenario is referred to as the regular demand and the third is called less-than-regular demand). The nominal demand associated with the regular demand scenario, in this case, is given by [80 80], while the nominal demand during inflationary scenario is marked by [120 120]. If the cumulative shortage exceeds a pre-defined limit, the demand corresponds to a less-than-regular scenario for which the nominal demand is given by [70 70]. This is shown in Figure 4.7 along with the probabilities of transition between the scenarios. While demand is in one of the regular, inflationary or less-than-regular scenario, it is also amenable to deviations from nominal demand. Deviations listed in Table 4.4 are considered along with probability transitions associated with matrix $T1$ (low uncertainty). The demand model considered here is like a

looped Markov chain. A realization of the model is shown in Figure 4.7. The model is similar to the random fluctuations around a seasonal mean, except that the mean switches are random and the probability of switching depends on actions/ performance. Resource life is considered to be 17 for the deterministic demand case. For the stochastic demand cases resource life is normally disctributed with mean 17 and variance 1.

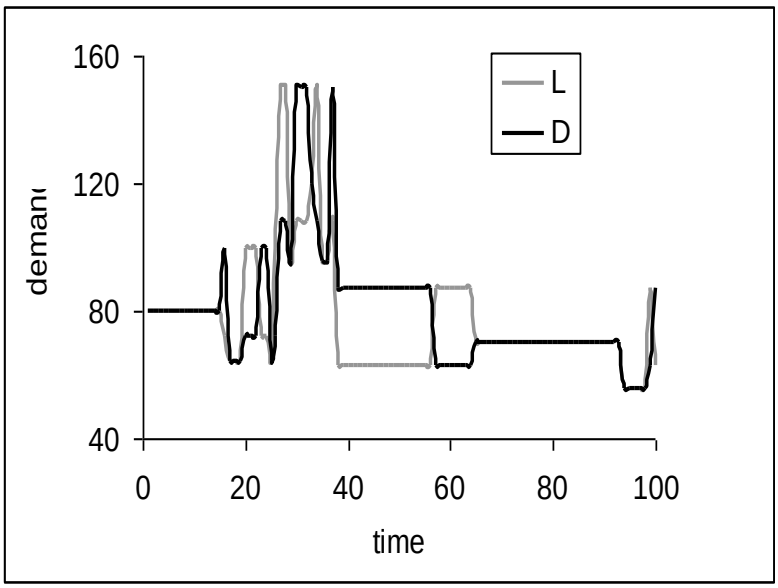


Figure 4.7. Sample realization of switching mean (inflationary demand scenario)

4.3.4 Results and discussion

The results obtained by using the ADP algorithm for three demand cases described in section 4.3.3 are shown in Table 4.5. The results from solving the LP for the deterministic demand and resource life and those from solving the moving horizon LP for the rest of the cases are also shown. For the deterministic case, LP gives the optimal result while the performance of ADP is suboptimal due to the value function

approximation, state aggregation and reduced state space. The infinite horizon discounted profit obtained from ADP is 94.5% of the optimal solution.

For the stochastic demand and resource life cases, however, the rolling horizon approach misses substantial amounts of demand. The behavior is expected since the rolling horizon method only considers the most expected demand trajectory for decision making at each time. When the demand deviations change, the actions recommended by the rolling horizon approach cannot cope with the modifications in demand. The behavior is more pronounced in the case of inflationary demand. This is because, in the rolling horizon approach, the cumulative shortage is almost always higher than the prescribed limit to sustain the regular demand. This is not the case in ADP approach, due to which a higher overall demand is observed, leading to higher sales and profit.

Table 4.5: Results from ADP and rolling horizon solution methods

	ADP			LP		
Demand case	**I**	**II**	**III**	**I**	**II**	**III**
total profit	529827.18	445917.47	429285.89	560310.30	409244.30	367832.29
total sales	16878.00	14205.00	13963.00	17939.80	13501.33	13006.21
total production	17042.00	14285.00	13763.00	17919.80	13481.33	13103.43
Average product inv	142.52	300.26	30.52	19.20	276.02	73.74
Average resource inv	11.65	19.93	7.91	9.68	21.93	10.67
total reorder	996.00	1006.00	1010.00	1013.99	908.77	906.48
total lost demand (L)	947.00	898.70	4208.00	2120.00	2228.64	4381.68
total lost demand (D)	1135.00	3594.80	3699.00	1810.20	2560.03	4382.11
total demand (L)	8940.00	10045.00	10704.00	8940.00	10045.00	10644.00
total_demand (D)	10020.00	8245.00	11166.00	10020.00	8245.00	11126.00
profit per unit of demand	27.94	24.38	19.63	29.55	22.38	16.90

4.4 Conclusions

An alternative approach to solving the planning and scheduling problems with perishable resources is presented in this chapter. Due to absence of uncertainty at the scheduling level, the decisions are determined by heuristics for resource transition and allocation. These heuristics are guided by the solution of LP which was formulated in Chapter 3. For the solution of the large sized planning problem, an approximate dynamic programming algorithm is presented. For the deterministic problem and for the parameter set considered, the solution obtained by the ADP algorithm is 94.5% of the optimal solution.

Two models are considered to account for uncertainty in demand and resource life. While resource life is drawn from a normal distribution, the demand is modeled to randomly deviate (i) from a constant mean and (ii) from a randomly switching mean. The probability of mean switching in case (ii) is dependent on current state and action. It is observed that for both the stochastic demand models, the ADP algorithm produces substantially better solutions as compared with the rolling horizon approach. This is because the form of the uncertainty cannot be successfully captured using the rolling horizon approach. Additionally when mean switching depends on the performance, poor decision-making has a more far reaching effect on overall performance.

CHAPTER 5

HANDLING DEFECT PROPAGATION IN SYSTEMS WITH STATIONARY EQUIPMENT AND COSTLY JOB INSPECTION

5.1 Introduction

Many manufacturing systems have large machines/ equipment that deteriorate randomly. Examples can be seen in auto-parts manufacturing, semi-conductor manufacturing, chemical process industry, etc. The effect of this deterioration is generally reflected in one or a combination of the following: lower yield, higher fraction of defective intermediates, higher operating or maintenance cost, or increased probability of complete failure of the equipment. The random deterioration in a single machine is often modeled as a Markov chain (Osaki 2002), where the equipment can be in one of N states at any time. State is designated as i=1,2,…,N, with 1 being the best state and the machine progressively degrading until it reaches an absorbing state N. The state N may characterize a completely failed state or a state of worst possible machine performance leading to least economically favorable production scenario. The states 1,2,…N are rarely known to the decision-maker and the machine may end up in state N (failed), without decision-maker's notice termed as 'silent failures' in (Ivy and Polak, 2005).

To keep the machine from ending up in a failed state, an optimal maintenance policy is needed. Typical decisions include renewal/ replacement (bring the machine back to the state 1), repair (bring it back to a relatively newer state), machine inspection (incur a cost to know the machine condition), or inspection of machine's output. The inspection is needed because the machine condition is not directly observable and may lead to perfect or imperfect knowledge of the condition depending on the quality of observations. Xiong et al, (2002) present a survey on replacement and repair policies for randomly deteriorating systems found in existing literature and industry. Typical policy structures are block replacement, age replacement, order replacement, failure limit policy, sequential preventive maintenance policy, and repair cost limit policy. A more rigorous approach, introduced by Girshick (1952) is to use a Partially Observable Markov

Decision Process (POMDP) framework to obtain an optimal repair and inspection policy. To this end, a survey of maintenance studies on single machine systems prone to stochastic degradation is given in Pierskalla and Voelker, (1976); Monahan (1982). Structural properties of the optimal value function and optimal policies are derived for many cases.

A case of particular interest is the one where the machine deterioration is reflected in increased production of defective jobs considered by Smallwood and Sondik (1973). The information about the machine condition may be known only by means of costly inspection of the machine output which relates probabilistically to the machine condition. The problem of costly job inspection is considered by Smallwood and Sondik, (1973); Ehrenfeld (1976); White (1979); Monahan (1980), and is referred to as the case with 'imperfect and incomplete observation'. Due to high cost of inspection, it may not be economically favorable to test every processed job. Such characteristics are prevalent in jobs requiring specialized testing for quality variables like electrical properties, radioactivity, product composition, uniformity, etc.

In most real world situations, a job undergoes a series of operations on multiple machines. Therefore, the notion of incomplete job inspection motivates the analysis of defect accumulation and propagation in systems with multiple operations and (or) multiple machines. This is because the untested defective intermediates would propagate through the system, until found defective in the final testing. Due to the possibility of accumulation of defective intermediates, job scheduling may also be affected by machine renewal and job inspection decisions. It should be noted that even if the inspection of all jobs was favorable, in the presence of inspection errors of type I (Lee and Unnikrishnan, 1998), defective jobs would be reported as non-defective and allowed to propagate through the system. Only error-free or perfect inspection is considered in this work but the analysis can be easily extended to type I errors. The above mentioned aspects of defect accumulation and propagation in systems with stochastically degrading machine(s) are addressed by considering two process flow topologies:

(i) a re-entrant flow system characterized by a job going through the same operation more than once

(ii) a hybrid flow system which is a combination of serial and re-entrant flow

Since the knowledge of the deterioration level of the machine(s) and the un-tested defective jobs is not completely available, the problems are formulated as POMDP. However, addition of scheduling decisions in the presence of partially observable states leads to fairly large size problems even for simple real world systems. Fortunately, recent research (Pineau et al, 2003; Smith and Simmons 2004; Spaan and Vlassis, 2005) in the area of approximate solution methods for large POMDPs proves helpful in this regard. A point based solution method called Perseus (Spaan and Vlassis, 2005) is used to solve the above-mentioned problems and the experimental results are reported. Comparison with prevailing periodic policies for maintenance and inspection are also presented.

5.2 System Description

In this work, discrete manufacturing systems with single or multiple machines are considered. The general characteristics of the system and modeling assumptions are as follows:

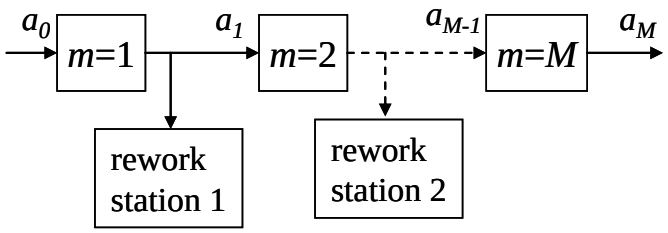
5.2.1 Modeling machine deterioration

All machines considered in subsequent problems are modeled to be deteriorating according to an underlying Markov chain. A good state is differentiated from a bad state by the associated probability of defect generation β_s, such that $\beta_s < \beta_{s+1}$ for s=1,2…N-1. Actions of machine renewal, job inspection and job scheduling are considered. The processed job is observed to be either defective or non-defective with complete accuracy whenever job inspection is performed. In case of multiple machine systems, the state transition probabilities and defect probabilities corresponding to machine states are independent from one machine to another unless otherwise mentioned.

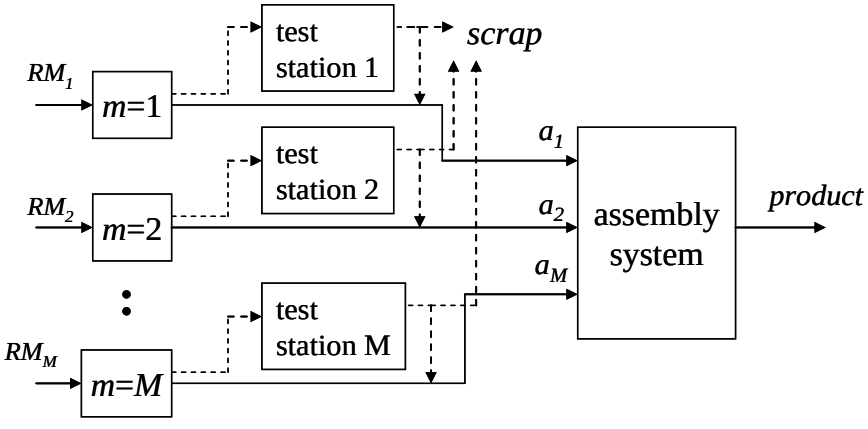
5.2.2 Defect accumulation and propagation

It is assumed that a defective job can be scrapped or reworked (depending on the

problem specification), only when a job inspection is carried out at that instant. If the job is not inspected due to economic reasons, the defective jobs tend to accumulate and propagate through the system. Figures 5.1(a) and 5.1(b) show a serial manufacturing system and a parallel assembly system respectively (Mandroli et al, 2006). The jobs (denoted by a_l for job completing the l^{th} operation) that are found defective can be reworked/ repaired in the serial manufacturing system, while defective jobs would be scrapped in the assembly system when found defective. The defective jobs that are not inspected would go on to the next operation or final assembly. It is assumed that when a job is inspected, defects caused by all prior operations are revealed as opposed to just the last operation.



(a)



(b)

Figure 5.1: (a). Serial production system with rework (b) Assembly system with scrap

For the purpose of modeling, the defective jobs in the system need to be kept track of at all time. Therefore, at any time, the system state can be fully characterized by two pieces of information: (i) the state of all the machines (ii) the total number of intermediates and the fraction of defective items in them. To differentiate the general system state from machine state, the latter is referred by machine *condition* or *deterioration level* in subsequent analysis.

5.2.3. Objective

Most studies on optimal maintenance policies for randomly deteriorating systems minimize the finite or infinite horizon cost (Smallwood and Sondik, 1973; Osaki 2002; Ivy and Polak, 2005). This is because the degradation of the machine is reflected in increasing operating cost and/or increasing maintenance cost as the machine regime gets worse. For example in (Ivy and Polak 2005), the cost of repair increases with the extent of repair, which in turn depends on how severe is the deterioration. In this work, it is assumed that the cost of renewal is the same for all machine regimes. Since inspection is carried out on jobs only, inspection cost is not a function of machine regime. Consequently, the deterioration is only reflected in the fraction of defective jobs, an increase in which leads to lower revenue. Therefore, the infinite horizon profit is maximized for all illustrations.

A good heuristic used in industrial applications is to employ an age replacement/renewal policy and periodic inspection policy. In similar spirit, heuristics of the following nature are used to establish a lower bound on the POMDP solution.

- maintain every θ_m time units
- test every θ_t time units

The best periodic policy also helps to obtain a sample set of belief points representing the relevant region for carrying out PERSEUS iterations. The FOMDP solution provides a loose and unachievable theoretical upper bound.

5.2.4. The single machine system

For a single machine prone to random degradation, the following cases have been extensively studied:

- Fully observable – the machine state is perfectly observable
- Unobservable – no information about the machine condition is available at any time
- Imperfect observation – imperfect observations e.g. information about processed job is readily available at all times
- Costly inspections – machine inspection or job inspection may be carried out at a cost.

For all of the above cases (Derman 1963; White 1979; Monahan 1980) have proven the existence of optimal control limit policies under certain assumptions on the system dynamics and reward function. A particular instance is the unobservable case where the optimal policy is to replace every m runs, where m can be infinity (Derman 1963). The conditions for such a policy to be optimal are:

(i) $r(s,a)$ is nonincreasing in $s \; \forall a$

(ii) For an ordering $a' > a$

$r(s,a') - r(s,a)$ is monotone in s

(iii) $\sum_{j=k}^{N} p_{ij}$ is nondecreasing in i for $\forall a, k \in \{1,2,..N\}$

In order to understand the concept of partial and incomplete observation and lay the foundation for future illustrations, an instance of the 'general repair and inspection model' presented in (Monahan 1982) is shown as illustration I. For ease of exposition, the transition probability matrix is considered to be an upper triangular matrix for all actions except that of machine renewal. This requires that p_{ij} =0 for $i<j$ and p_{NN} = 1, making $s=N$ an absorbing state. The transition probabilities corresponding to the renewal action have p_{i1} =1, p_{ij}=0, $j \neq 1$.

Illustration I: A hypothetical machine produces one job per unit time and is prone to deterioration according to the model described earlier in this section. Pertinent decisions include machine renewal and job inspection, both of which are assumed to be instantaneous and have associated costs C_M and C_I respectively. The machine may transition to a different deterioration level at each time. Degradation time-scales are therefore controlled by the probability transition matrix corresponding to the action(s) of

non-renewal. A reward C_P is received only if the processed job is non-defective (which is determined for all jobs during final testing before product sale).

$S = \{1,2,.. ,N\}$

$A = \{a_1,a_2,a_3\}$: a_1 - do nothing; a_2 - inspect; a_3 - renew

$O = \{o_1,o_2,o_3\}$: o_1- no defect; o_2 - defect; o_3 - no observation

$R(s,a,s') = (1-\beta_{s'})C_P - I_M(a)C_M - I_I(a)C_I$

$O_{a2}(o_1|s') = 1-\beta_{s'}$; $O_{a2}(o_2|s') = \beta_{s'}$

$O_a(o_1|s') = 1$ for $a \neq a_2$

where, $I_M(a)$ and $I_I(a)$ are binary numbers equal to 1 when the machine is renewed and job inspection is performed respectively. The POMDP for N=3 (three levels of deterioration) with C_P = 1000, C_M = 10000 and three different values of C_I (parameter sets 1,2 and 3 shown in Table 5.1), is solved using PERSEUS (as discussed in Section 2.3 of Chapter 2) and the optimal policy structure is shown in Figure 5.2 (for C_I =150). It is seen that the optimal policy has a *control limit* structure due to the following system properties.

(i) $\quad r(s,a) - r(s1,a) \geq 0 \quad for \;\; s < s1$

(ii) $\quad p(s'|\, s,a) - p(s'|\, N,a) \geq 0 \quad for\, s' \neq N$

(iii) $\quad p(o_1 \mid s,a) - p(o_1 \mid s1,a) \geq 0 \quad for\, s < s1$

(iv) $\quad p(o_1 \mid s=1,a) \geq p(o_2 \mid s=1,a) \;\; \forall a$

The policy can therefore be compactly represented as (5.1) for the above parameter values. It can be shown that a general system with N-state deterioration and above properties satisfies the monotonicity properties shown by (Monahan 1982). Those noted above represent sufficient conditions for monotonicity results to hold. This is demonstrated in appendices A through C.

$$policy = \begin{cases} do\ nothing & if\ b(3) \leq 0.09 \\ renew \quad if & \frac{4}{3}b(3) - 2b(1) \geq 1 \\ inspect\ job & otherwise \end{cases} \qquad (5.1)$$

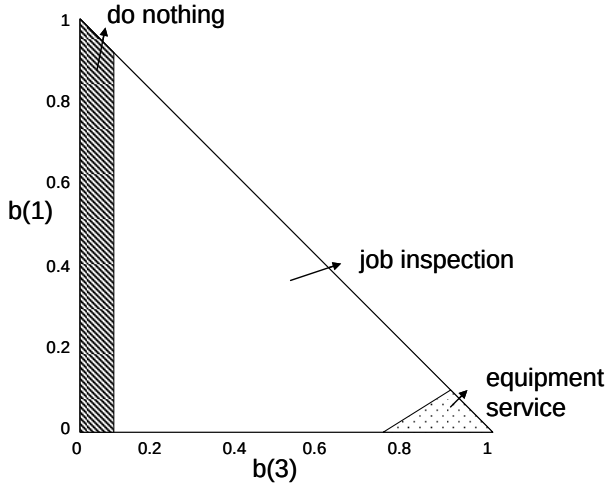


Figure 5.2: Optimal policy for three-state single machine problem

In the illustrations to follow, the concept of imperfect and incomplete observation is extended to multiple type of jobs operated on a single machine (illustration 2) and finally to multiple type of jobs operated on multiple machines (illustration 3). In both examples, propagation of defective jobs contributes most to the problem size and computational complexity.

5.3 A re-entrant flow example - modeling and solution

5.3.1 Description

Illustration 2: The machine in illustration 1 again operates on one job per time unit and undergoes degradation at each time, according to the Markov chain similar to the one in illustration 1. However, the job cycles back to the machine until it undergoes the same operation L times, after which it leaves the system as product. (This re-entrant characteristic is observed in semi-conductor fabrication where multiple layers are deposited on silicon wafers. Therefore, jobs at various stages of production compete for the same resources). The process is shown in Figure 5.3, where $\tilde{a}_l$ refers to the job and subscript l=0,1…L refers to the number of operations that the job has gone through. For

simplicity they are called *l layers*. There is a queue before the operation where intermediate jobs wait for processing. Therefore, an added decision in this case is job scheduling, i.e., which of the intermediates $\tilde{a}_l$,l=0,1,…L-1 to admit for processing. Each intermediate can be inspected if the decision-maker so chooses. Defect in all existing layers can be detected at the time of inspection. If found defective, the intermediate job is immediately scrapped/ removed from the system. But if the inspection is not carried out at each time, then defective items would propagate through the system. The product brings revenue C_P only if all the L layers are non-defective. The costs for machine renewal, job inspection, processing l^{th} layer and raw material $\tilde{a}_0$ are C_M, C_I, C_l and C_0 respectively. The overall objective motivated by quality management is to devise an optimal machine renewal, job inspection and job scheduling policy that maximizes the infinite horizon profit from product sales. It is assumed that supply of $\tilde{a}_0$ is unlimited and final product $\tilde{a}_L$ is always tested. (The symbol $\tilde{a}$ is used to differentiate the job from action a. The nomenclature with *tilda* (~) is maintained to denote jobs and machines in subsequent analysis, in order to avoid confusion with variables related to state and action spaces).

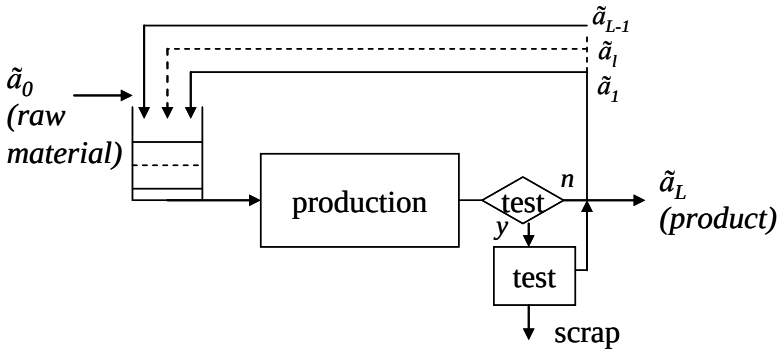


Figure 5.3: Re-entrant flow problem

The above problem is interesting in the following ways:

i. It allows for analysis of the propagation and accumulation of defective jobs by means of a compact system representation.
ii. For very small and very large queue sizes, the system would behave as a serial production (Figure 5.1(a)) and assembly system (Figure 5.1(b)) respectively. For example, when no jobs are allowed to wait in the queue, one job remains in the system until completion. This is similar to the job going through a sequence of L operations in a series. On the other hand, if a large number of intermediates are waiting in the queue for processing, then it acts more like an assembly system.
iii. Job inspection now serves two purposes, i.e., it not only provides information about the machine degradation, but also gives information about defective intermediates so that they can be picked out of the system to save the cost of additional operation on them.
iv. The job gathers value with each deposited layer. With better inspection and job scheduling, it is possible to reduce the number of good layers lost on bad products. This is because, a product is considered defective if at least one layer on it is defective.

Similar to the previous illustration, the system is modeled as a POMDP. The modeling details are included in the formulation and examples are presented for a three layer product, i.e. for L=3. The problem is referred to as the *re-entrant flow* problem. It is worth noticing that the state dimension and consequently, the size of the state space are significantly larger in this case. This is because now the total number of intermediates in the system and the fraction of defective intermediates need to be accounted for. The action space has an added dimension of job scheduling and the state transition probability matrix also takes into account the probability of defect propagation.

5.3.2 Formulation as POMDP

State

The system at any time is fully characterized by the total number of jobs, the fraction of defective jobs and the deterioration level of the machine. Therefore,

$s = [n_1\ n_2\ ...n_{L-1}\ d_1\ d_2\ ...d_{L-1}\ \Gamma\ ]$

$n_l \in \{0,1,2..,\eta\}$- total number of $\tilde{a}_l$ in the queue for l=1,2..L-1

$d_l \in \{0,1,2..,n_l\}$ - number of defective $\tilde{a}_l$ in the queue for l=1,2..L-1

$\Gamma \in \{1,2..,N\}$- discrete integer representing the deterioration level of the machine

The state space consists of all possible combinations of the above parameters. For instance, if L=3 and if the maximum allowable number of jobs in the queue (η) is limited to 3 (i.e., $n_1 + n_2 \leq 3$), then there are following (n_1,n_2) combinations: (3,0) (2,1) (1,2) (0,3) (2,0) (1,1) (0,2) (1,0) (0,1) (0,0). For a particular value of n_1 say 3, d_1 can hold four possible values from 0,1,2,3. Therefore, the total number of possible combinations for [n_1 n_2 d_1 d_2] is 35. With 3 deterioration levels for the machine, the size of the state space is 105 (35 × 3). Similarly, the size of the state space for maximum queue sizes of 4 and 5 are 210 and 378 respectively.

Action/ Decision

$a = [\ a_1\ a_2\ a_3]$

where $a_1 \in \{0,1,2,..L\text{-}1\}$ pertains to the job scheduling decision (admit $\tilde{a}_0$, $\tilde{a}_1$ … $\tilde{a}_{L-1}$) ; $a_2 \in \{0,1\}$ pertains to job inspection decision (test (1) the processed job or not (0)) ; $a_3 \in \{0,1\}$ pertains to renewal decision (renew the machine(1) or not (0)). Assuming all final products are tested, the size of action space for L=3 is ((3 x 2)-1) x 2 = 10.

Observation

$o \in \{o_1,o_2,o_3\}$

Transition and observation probability matrices

For a queue length of 3, T is a 105 × 105 × 10 matrix incorporating the 3 sources of uncertainty mentioned below:

a). Machine regime switching - As shown in illustration 1, the machine can switch between regimes with certain probabilities in a non-deterministic manner.

b). Defect generation - Defect generation is probabilistic and the defect probability (β_s) is set by the regime in which the machine is operating.

c). Error propagation- Since not all intermediates are tested, the queue can contain defective intermediates, designated as d_1 and d_2 in the state description. Probability that a defective intermediate is picked and operated upon is given by q:

$$q = \frac{d_l}{n_l} \qquad \text{For } \tilde{a}_l \text{ being operated}$$

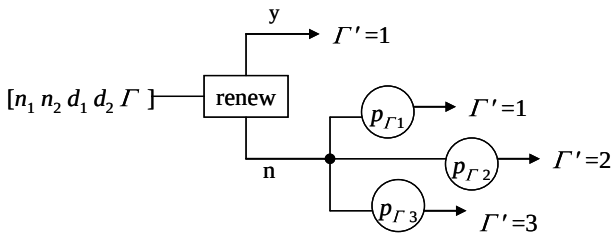
For a queue length of 3, O is a 105 × 10 × 3 matrix. It must be noted that the total number of the intermediates in the sysem (n_1 n_2 ...n_{L-1}) are always observable. The specific form of state transitions for three levels of machine deterioration, L=3 and a_1=1 is shown in Figure 5.4. If n_l = 0, admitting $\tilde{a}_l$ for processing is not a permissible action. To avoid this situation while implementing the POMDP policy, $\tilde{a}_{l-1}$ is admitted.

Objective

The infinite horizon discounted profit/ reward is given by (5.2):

$$V^{\pi^*} = \max_{a_1, a_2 \ldots a_\infty} \sum_{t=1}^{\infty} \gamma^{t-1} (C_P I_P(a_t) - C_M I_M(a_t) - C_I I_I(a_t) - C_l I_l(a_t) - C_0 I_0(a_t)) \tag{5.2}$$

where γ is the discounting factor, a_t is the action at time t and all I's (I_P, I_M, I_I, I_l, I_0) are binary and are equal to 1 when a non-defective product is produced, when a maintenance job is run, when an intermediate job is tested when $\tilde{a}_l$ is run and when raw material $\tilde{a}_0$ is admitted at time t, respectively.

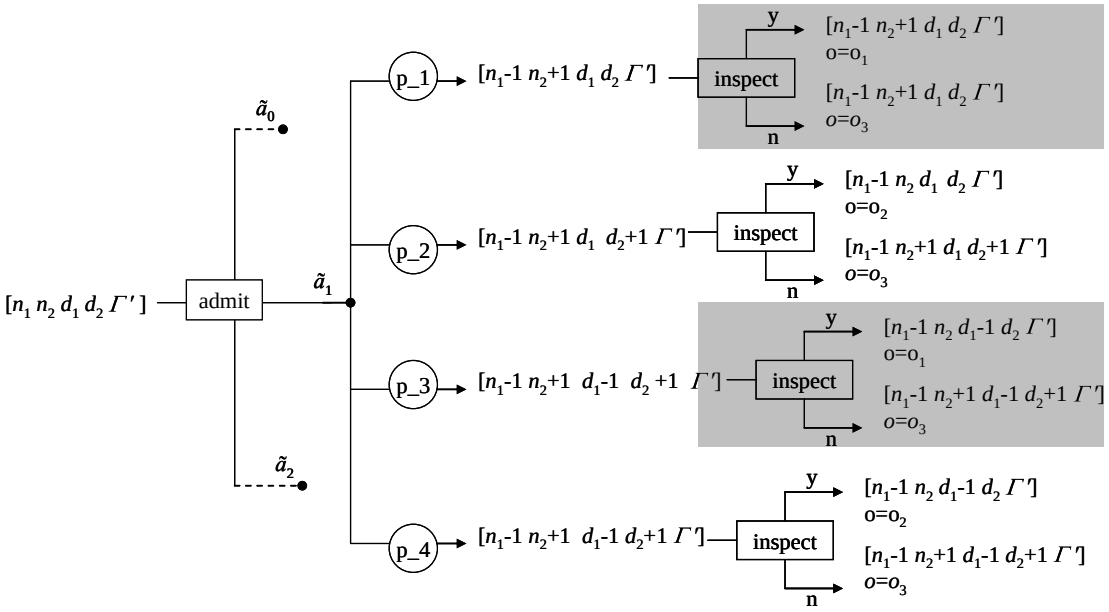


(a)

$$p\_1 = (1-\beta_\Gamma)(1-\frac{d_1}{n_1});\ \ p\_2 = \beta_\Gamma(1-\frac{d_1}{n_1});$$

$$p\_3 = (1-\beta_\Gamma)\frac{d_1}{n_1};\ \ p\_4 = \beta_\Gamma\frac{d_1}{n_1}$$

(b)



(c)

Figure 5.4: State transition for the re-entrant flow problem for 3 levels of machine deterioration and L=3 (a) Possible values of the machine condition at the next time step (Γ') given the current machine condition (Γ) and the renewal decision. (b) Probabilities associated with defect generation (β_Γ) and propagation (d_1/n_1). (c) Possible next states and observations depending upon job scheduling and job inspection decisions and realization of uncertainty.

The maximum allowable size of the queue largely governs the size of the state space which eventually controls the size of the problem. The above problem is solved for maximum queue lengths of 3, 4 and 5. Three different parameter sets (4, 5 and 6) shown in Table 5.1(a) are considered.

The parameter values are reasonably chosen to represent the trade-offs among different cost heads in a typical manufacturing environment. Since the queue length is constrained, holding cost/ work in progress (WIP) cost is not considered. All problem instances are solved using the algorithm shown in Figure 5.5. An initial belief sample set is obtained by using three different policies (i) optimal policy for the FOMDP problem, (ii) best periodic or block replacement and inspection policy, and (iii) policy to select a random action at each time. The algorithm uses PERSEUS iterations on the fixed initial belief set until ε convergence is achieved. A new sample is then obtained using the current value function for POMDP and the above is repeated (this is called one *sample iteration*). These sample iterations are carried out until the performance of two subsequent sample iterations is found to be δ-close for a randomly chosen test belief set.

The results are reported in Table 5.2. Table 5.2 includes the size of the problem for the queue sizes considered. $|V|$ is the size of the optimal policy, i.e., the number of gradient vectors α_n^i, i= 1, 2, .., $|V_n|$. Profit for POMDP is the average profit obtained by starting in s=1 ([0 0 0 0 1] -no jobs in the system and best machine condition) and following the optimal policy. Average is taken over 100 experiments in all cases. FOMDP profit reported for starting in state s=1, acts as the theoretical upper bound and cannot be achieved. The difference in the two values provides the extent to which the partial observability affects the performance. For parameter set 5, the processing costs for layers 1,2 and 3 are higher as compared to those for parameter set 4. This causes reduction in the overall profit as compared with parameter set 4 but evidently no difference in the optimal policy for fully observable problems. However, in the partially observable case, the (near) optimal policy results in increase in average queue size (number of intermediates in the queue at any time) with increase in processing cost. This is because the system is more cautious about running an expensive intermediate when

machine deterioration level is high. In parameter set 6, the defect probabilities associated with machine deterioration levels 2 and 3 are increased, which leads to further reduction in overall profit. The general characteristics of FOMDP policies are discussed in further details.

5.3.3 Characterization of FOMDP policy

In order to understand the system behavior, the optimal policy corresponding to the FOMDP problem is analyzed. It is seen that the machine renewal, job inspection and job scheduling decisions are mutually correlated and therefore a compact representation of the policy is not possible. The general characteristics of the optimal policy are discussed further:

1. The optimal policy is a strong function of the probability of defect *generation* and that of defect *propagation*. The former is the determined by β_s, the defect probability associated with machine deterioration level s and the latter is the probability that the incoming job is already defective. The latter is given by d_l/n_l. Also the term 'expensive intermediate' is used to denote a_l with relatively large l.
2. The machine is renewed when defect generation probability is high (3 and(or) 2) and defect propagation probability is low.
3. Job inspection is carried out when both of the above probabilities are high and when an expensive intermediate is admitted for processing. Note that according to problem specification, $\tilde{a}_L$ is always tested.
4. An expensive intermediate $\tilde{a}_l$ is admitted for processing whenever $n_l \neq 0$ and defect generation and propagation probabilities are low. Otherwise, $a_{l\text{-}1}$ is picked for processing.

For the cost values considered, the system tends to keep a small number of intermediates in the system as guided by the optimal policy. This is the reason why the optimal policy and the performance of the reentrant flow problems with varying limits on queue sizes (3,4 and 5) are the same (please see Table 5.2). As for the structure of the POMDP policy, trends similar to the FOMDP policy are observed. However, job

inspection also serves the purpose of determining machine condition which is not known with certainty along with the fraction of defective intermediates. The policy space is very large in the case of POMDP problem to be represented in a meaningful way. This is because the optimal policy is a map between the high dimensional belief state to a relatively small set of actions. Since the policy is characterized by the value function, some conjectures on the structure of the value function for re-entrant flow problem are presented in section 5.3. The case with multiple machines in the *hybrid-flow* example is presented below. It combines the re-entrant flow feature with serial flow topology as shown further.

Step 0. Generate a sample belief B with 10,000 belief points by simulating the system under the policies below:

(i) FOMDP optimal policy assuming full observability

(ii) Various periodic maintenance and inspection policies

(iii) Random actions

Initialize $V^{init} = \min_a\{\min_s r(s,a)\}$

Step 1. Using B and V^{init}, run PERSEUS iterations as shown in section 2.3 of Chapter 2 for ε = 0.01. The converged value function is denoted by $V^*_{PERSEUS}$

Step 2. Use $V^*_{PERSEUS}$ to sample another belief set $\hat{B}$ with 10,000 states.

Step 3. Set $V^0 = V^*_{PERSEUS}$. Make one PERSEUS iteration to obtain V^1

Step 4. If $|V^1 - V^0|_\infty \le 0.01$, stop, else set $B = \hat{B}$, $V^{init} = V^1$, go to Step 1.

Figure 5.5. Algorithm I for solving POMDP

5.4 A hybrid flow example – modeling and solution

5.4.1 System Description

Illustration 3: There are three machines (*A*,*B*,*C*) similar to the one in Illustration 1 that undergo degradation according to separate independent Markov chains and defect probabilities. The machines are in series and the jobs have a pre-defined order of operation as shown below:

1. Three layers at machine $\tilde{A}$
2. Two layers at machine $\tilde{B}$
3. One layer at machine $\tilde{C}$

where *layer* again refers to one machine operation for simplicity. The jobs being operated at machines $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are designated as $\tilde{a}_l$, $\tilde{b}_l$, and $\tilde{c}_l$ respectively, where the subscript l refers to the number of layers already deposited. The process is schematically shown in Figure 5.6. Due to the difference in times of operation, the machines must deteriorate after each run in this case. However, for simplicity, it is assumed that all machines can transition to a lower deterioration level at each time unit but the result is only reflected on the next job to be processed and not the current job. There is an inspection station after machines $\tilde{A}$ and $\tilde{B}$. If an intermediate is tested and found defective, it is sent to a repair station where only the topmost layer can be repaired. It costs C_R for each repair and the repaired job is returned to the system for further processing. It takes 2 time units for an operation at $\tilde{A}$, 3 units at $\tilde{B}$ and 6 units at $\tilde{C}$. Unlike in Illustration 2, there is no possibility of queuing the jobs in this system. Jobs are fed sequentially, there is one job at each machine at any time and product is obtained every 6 time units. It is assumed that machine maintenance, job inspection and rework take negligible time. However, if a defect is detected in the last layer of $\tilde{a}$ or $\tilde{b}$, then repair is not required and it can be repaired by the subsequent operation ($\tilde{B}$ or $\tilde{C}$ resp.) without any additional cost. Reward is received only when all the layers in the final product $\tilde{c}_1$ are non-defective. The objective is to maximize the average infinite horizon discounted profit while obtaining an optimal renewal policy and job inspection policy for all three operations. It is assumed that the final product is always tested.

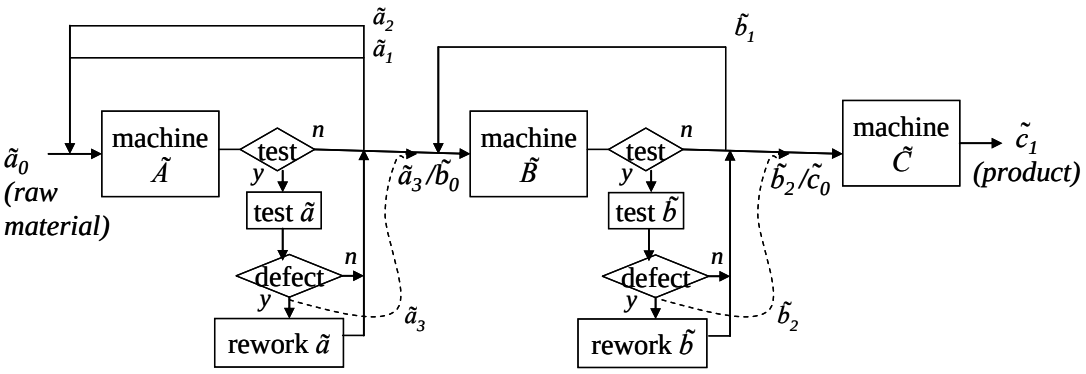


Figure 5.6 Hybrid –flow system

The feature of the above problem that a defect in the last layer of $\tilde{a}$ and $\tilde{b}$ can be corrected by subsequent operations is seen in automotive assembly where downstream correction of errors in physical dimensions of jobs is possible. Due to this feature, the maintenance decisions downstream affect the upstream processing. The system is balanced since a job spends exactly 6 time units at each machine. A time counter t (t=1,2..,6) is used to designate the time elapsed since the job first entered the machine. It is assumed that the machine can be serviced and job can be inspected only at the end of a run. Therefore, maintenance and inspection at machine $\tilde{A}$,$\tilde{B}$ and $\tilde{C}$ can be done when t is a multiple of 2, 3 and 6 respectively. The formulation of the problem as a POMDP is presented below.

5.4.2 Formulation as POMDP

State

The system at any time is fully characterized by the following

$s = [\Gamma_{\tilde{A}}\ \Gamma_{\tilde{B}}\ \Gamma_{\tilde{C}}\ t\ defect_{\tilde{a}}\ defect_{\tilde{b}}\ defect_{\tilde{c}}]$

where,

$\Gamma_{\tilde{A}}, \Gamma_{\tilde{B}}, \Gamma_{\tilde{C}} \in \{1,2,..N\}$ represent the regime of machines $\tilde{A}$,$\tilde{B}$ and $\tilde{C}$

$t1 \in \{1,2,..,6\}$ time elapsed until the jobs $\tilde{a},\tilde{b},\tilde{c}$ started getting processed at machine $\tilde{A},\tilde{B}$ and $\tilde{C}$ respectively.

defect $\in \{0,1\}$ shows whether the jobs $\tilde{a},\tilde{b},\tilde{c}$ that are being processed have one or more defective layer(s) (1) or not (0)

Action/ decision

$a = [renew_{\tilde{A}}\ \ renew_{\tilde{B}}\ \ renew_{\tilde{C}}\ \ test_{\tilde{a}}\ \ test_{\tilde{b}}]$

$renew_i \in \{0,1\}$ for $i = \tilde{A},\tilde{B},\tilde{C}$, pertains to whether to renew the machine i (1) or not (0)

$test_i \in \{0,1\}$ for $i = \tilde{a},\tilde{b}$ pertains to whether to test the processed job i(1) or not (0)

Observation

$o = [o_{\tilde{a}}\ \ o_{\tilde{b}}\ \ o_{\tilde{c}}]$

where $o_i \in \{o_1,o_2,o_3\}$ for $i = \tilde{a},\tilde{b},$

$o_i \in \{o_1,o_2\}$ for $i = \tilde{c}$

This is because the final product is always tested

Transition Probability Matrix

T incorporates the following sources of uncertainty

a). Machine regime switching - As shown in illustration 1 (Figure 5.4(a)), the machines can switch between regimes with certain probabilities in a non-deterministic manner.
b). Defect generation - Defect generation is probabilistic and the defect probability (β_s) is set by the regime in which the machines are operating.

Objective

Maximization of the infinite horizon discounted profit/ reward given by (5.3):

$$V^{\pi^*} = \max_{a_1, a_2, .. a_\infty} \sum_{t=1}^{\infty} \gamma^{t-1} \{ C_P I_P(a_t) - \sum_{i=\tilde{A}}^{\tilde{C}} C_M^i I_M^i(a_t) - \sum_{j=\tilde{a}}^{\tilde{b}} C_I^j I_I^j(a_t) - C_0 I_0(a_t) \} \quad (5.3)$$

where,

γ is the discounting factor, a_t is the action at time t and all I's (I_P, I_M, I_I, I_0) are binary and are equal to 1 when a non-defective product is produced, when a maintenance job is run, when an intermediate job is tested, and when raw material $\tilde{a}_0$ is admitted at time t, respectively.

The above is solved for two possible regimes (N=2) for each machine and the parameter sets 7,8 and 9 shown in Table 5.1. The POMDP is solved using the algorithm shown in Figure 5.5 and the results are reported in Table 5.2. Similar to the results for illustration 2, the FOMDP profit is also reported to highlight the extent of partial observability in each case. Similar to the re-entrant flow case, the policy space is complicated leading to difficulties with compact policy representations even for the fully observable problem. For the parameter sets 7, 8 and 9 shown in Table 5.1, the characteristics of the optimal FOMDP policy are as follows:

1. When the time counter $t1$=2, only machine $\tilde{A}$ can be renewed and $\tilde{a}_1$ inspected. The optimal policy pertaining to all parameter sets (7,8 and 9) is to never renew the machine and always inspect the job $\tilde{a}_1$.
2. When $t1$=3, the optimal policy is never to renew machine $\tilde{B}$ and always inspect job $\tilde{b}_1$
3. When $t1$=4, the optimal policy is to not renew machine A but to inspect job $\tilde{a}_2$ only when $\tilde{a}_1$ is non-defective. This is expected since only the top layer can be repaired upon inspection. However since all $\tilde{a}_1$s are tested at $t1$=2, this situation never arises in the fully observable case.
4. When $t1$=6, only machine $\tilde{C}$ is renewed when machine is in deterioration level 2 and the incoming job is non-defective, for parameter sets 7 and 9. For parameter set 8, machine $\tilde{A}$ is also renewed when in deterioration level 2. This difference can be attributed to lower renewal cost in case of parameter set 8.

The high dimensionality associated with the (near) optimal policy for the partially observed hybrid flow problem prevents a compact representation. Some conjectures on the POMDP policy together with those on the partially observed reentrant flow problem are presented in the following section. Alternative policies are also discussed in order to establish the goodness of the POMDP solution.

5.5 Discussion on results and policy discussion

5.5.1 Performance Comparison

In order to understand the advantages of a rigorous approach to solving this class of problems, the following is used as a basis of comparison:

(i) FOMDP solution – The performace of the MDP, assuming that the system state is fully observed, establishes a non-achievable upper bound to the POMDP solution and the gap between the performances show the extent to which the partial observability affects the system. It also helps understand the policy structure and the relevant region of the state space in certain cases. The optimal discounted infinite horizon reward for starting in s=1 for all illustrations and parameter sets is reported in Table 5.2. For the single machine problem, the changing inspection cost has no effect on the solution since the state is fully observed and inspection is never carried out. (State s=1 in all illustrations, represents the starting state with the best machine regime(s) and no jobs in the system).

(ii) Q^{MDP} approximation – A lower bound on the close-to-optimal solution of the POMDP is established by using a simple function approximation scheme (Hauskretch 2000). The optimal Q-function associated with the fully observable MDP is shown in (5.4), where V^{MDP} is the optimal value function. The Q-function associated with the partially observed problem for each belief state and action is then approximated as shown in (5.5). The resulting performance is contained in Table 5.2. Note that the optimal policy corresponding to the FOMDP does not contain the inspection decision for gauging machine regime when states are fully observed. That is the reason why, at

times, the performance of this approximation is worse than that of periodic policy as discussed next.

$$Q^{MDP}(s,a) = r(s,a) + \sum_{s'} p(s'|s,a)V^{MDP}(s) \tag{5.4}$$

$$\widehat{Q}(b,a) = \sum_{s=1}^{N} Q^{MDP}(s,a)b(s) \tag{5.5}$$

Table 5.1: Parameter values for (a) the re-entrant flow problem (b) the hybrid flow problem (-do- implies ditto)

parameter set number	C_P	C_M	C_0	C_I	$\tilde{a}_0$	C_I $\tilde{a}_1$	$\tilde{a}_2$	C_R	machine regime transition	beta
1	1000	10000	-	150		-		-	$\begin{bmatrix} 0.99 & 0.0075 & 0.0025 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{bmatrix}$	$[0.05 \quad 0.1 \quad 0.5]$
2	1000	10000	-	50		-		-	-do-	-do-
3	1000	10000	-	500		-		-	-do-	-do-
4	2000	10000	100	150	20	100	200	-	$\begin{bmatrix} 0.99 & 0.0075 & 0.0025 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{bmatrix}$	$[0.05 \quad 0.1 \quad 0.5]$
5	2000	10000	100	150	100	200	500	-	-do-	-do-
6	2000	10000	100	150	100	200	500	-	-do-	$[0.05 \quad 0.25 \quad 0.75]$

(a)

parameter set	C_P	C_M $\tilde{A}$	$\tilde{B}$	$\tilde{C}$	C_I $\tilde{a}$	$\tilde{b}$	C_R $\tilde{a}$	$\tilde{b}$	machine regime transition $(\tilde{A},\tilde{B},\tilde{C})$	$\beta(s)$
7	1000	2000	2000	3000	25	25	175	175	$\begin{bmatrix} 0.99 & 0.01 \\ 0 & 1 \end{bmatrix}$	$[0.1 \quad 0.6]$ $(\tilde{A})$ $[0.25 \quad 0.75]$ $(\tilde{B})$ $[0 \quad 0.5]$ $(\tilde{C})$
8	1000	1000	1500	3000	25	25	175	175	-do-	$[0.25 \quad 0.75]$ $(\tilde{A})$ $[0.25 \quad 0.75]$ $(\tilde{B})$ $[0 \quad 0.5]$ $(\tilde{C})$
9	1000	2000	2000	3000	25	25	175	275	-do-	$[0.1 \quad 0.6]$ $(\tilde{A})$ $[0.25 \quad 0.75]$ $(\tilde{B})$ $[0 \quad 0.5]$ $(\tilde{C})$

(b)

Table 5.2: Results for the re-entrant and hybrid flow problems

problem	$\|S\|$	$\|A\|$	$\|O\|$	*parameter set*	$\|V\|$	*FOMDP profit, s=1* (x 10^4)	*POMDP* profit, s=1* (x 10^4)	*periodic heu, s=1* (x 10^4)	*QMDP approx, s=1* (x 10^4)	*profit with linear V,s=1* (x 10^4)	*size of decision tree*
single machine	3	4	3	1	73	8.90 ± 0.39	8.49 ± 0.49	8.38 ± 0.89	8.04 ± 1.12	8.44 ± 0.58	4
				2	81	8.90 ± 0.39	8.61 ± 0.38	8.38 ± 0.89	8.09 ± 1.17	8.54 ± 0.44	
				3	84	8.90 ± 0.39	8.41 ± 0.20	8.38 ± 0.89	8.09 ± 1.17	8.21 ± 0.26	
re-entrant flow (queue =3)	105	10	30	4	423	3.19 ± 0.64	2.98 ± 0.45	2.45 ± 0.16	2.86 ± 0.57	2.96 ± 0.61	15
				5	442	1.64 ± 0.48	1.35 ± 0.23	0.78 ± 0.54	0.627± 0.8	1.37 ± 0.46	
				6	360	1.30 ± 0.68	0.99 ± 0.08	0.57 ± 0.74	0.92 ± 0.70	0.97 ± 0.28	
re-entrant flow (queue =4)	210	10	45	4	390	3.20 ± 0.61	2.91 ± 0.54	2.53 ± 0.21	2.75 ± 0.75		
				5	387	1.66 ± 0.61	1.35 ± 0.23	0.91 ± 0.45	1.09 ± 0.83		
				6	365	1.33 ± 0.75	1.00 ± 0.34	0.78 ± 0.63	0.68 ± 0.85		
re-entrant flow (queue =5)	378	10	63	4	300	3.20 ± 0.51	2.95 ± 0.71	2.80 ± 0.19	2.84 ± 0.85		
				5	292	1.66 ± 0.61	1.35 ± 0.23	0.81 ± 0.44	0.90 ± 0.73		
				6	383	1.36 ± 0.75	1.00 ± 0.34	0.62 ± 0.64	0.72 ± 0.65		
hybrid-flow system	352	32	27	7	83	1.11 ± 0.65	1.00 ± 0.46	0.40 ± 0.12	0.34 ± 0.12	0.50 ± 0.16	19
				9	83	1.09 ± 0.34	0.91 ± 0.15	0.55 ± 0.18	0.45 ± 0.18	0.71 ± 0.22	
				10	74	1.05 ± 0.23	0.88 ± 0.10	0.73 ± 0.18	0.71 ± 0.31	0.45 ± 0.49	

(iii) Periodic maintenance and (or) inspection policies – As mentioned in section 5.2, the periodic policies are easy to implement and form the industrial standard for maintenance and job inspection decisions. For single machine problem, a periodic maintenance policy is optimal when inspection costs are prohibitively high (shown as the unobservable case in section 5.2). As seen in Table 5.2, the performance of the POMDP for the single machine problem drops with increasing cost of inspection. For parameter set 3, the periodic policy gives a performance similar to that of the POMDP.

5.5.2 Empirical findings and conjecture

In addition to the rigorous results, the following empirical observations are reported for the illustrations that were studied:

(i) In the relevant belief space, the close-to-optimal value function for the partially observed re-entrant flow problem and for the single machine could be represented as a linear function of the belief states.

(ii) In the relevant belief space, the close-to-optimal decision rule for single machine, reentrant flow and hybrid flow could be represented as a decision tree of size substantially smaller than the dimension of the belief space.

The value function in this case can be claimed as only close-to-optimal because there are no guarantees for optimality of solutions yielded by PERSEUS in solving large size POMDPs. Relevant belief space refers to the set of belief points that are visited by following the close-to-optimal policy. Figure 5.7 is a plot between the actual value function v/s that obtained by a linear regression for the re-entrant flow case and parameter set 4. The value function is plotted for the belief states that are visited when POMDP close-to-optimal policy is followed. The band around 45° line represents a good fit. It is seen that the points mostly lie within that band.

5.5.3 Value function approximation

It is well-known that for a general infinite horizon POMDP, the optimal value function can be closely approximated as a piecewise linear and convex function (Sondik

1978) as shown in (2.10). From the finding in (i) above, it turns out that in the relevant region of the belief space, the value function can be approximated as a single linear function shown in (5.6) where w_s are the weights for each system state.

$$\text{s.t. } \tilde{V}(b) = \sum_{s=1}^{|S|} w_s b(s) \tag{5.6}$$

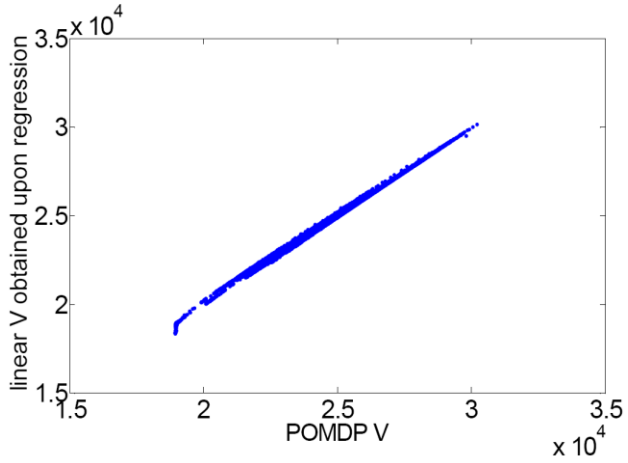


Figure 5.7: Value function v/s linear approximation plot

Therefore, the close to optimal value function can be represented as a set of weights $\{w_1, w_2..w_{|S|}\}$. In order to determine these weights, value iterations can be carried out on $|S|$ different belief points where $|S|$ is the dimension of the belief state. In Figure 5.8, an algorithm to solve the POMDP in such a scenario is presented. The results from algorithm II are also reported in Table 5.2. It is seen that the performance for the single machine and re-entrant flow cases are comparable with that of algorithm I.

Step 0. Arbitrarily initialize W^0, where W is the vector of w_i, for i=1,2,…|S|. Set $N = |S|$ and $B = I_N$, where I_N is the identity matrix of size N

Step 1. Given $V(b) = bW^0 \ \forall b$, run value iteration (equation 4) for belief set B, until $|V^{i+1} - V^i|_\infty \leq 0.01$. Denote resulting V as V_B^*

Step 2. Determine $W^1 = (B^T B)^{-1} B^T V_B^*$

Step 3. Use W^1 to sample $\hat{B}$ with 10,000 belief points such that *rank*($\hat{B}$) $\geq N$.

Step 4. Run one value iteration step on $\hat{B}$ to obtain V_{Bbar}.

Step 5. If $|V_{\hat{B}} - V_B^*|_\infty \leq 0.01$, then stop, else set $B = \hat{B}$, $W^0 = W^1$ and go to Step

Figure 5.8: Algorithm II to solve POMDP with linear value function approximation.

5.5.4 Decision-tree analysis

A decision tree serves as a good tool to represent a policy. A simple decision tree for the optimal policy for the single machine problem is shown in Figure 5.9. The solid circles represent a condition on belief state and the branches show the different actions associated with it. The size of the decision-tree is determined by the number of levels at which such conditions are posed. For the example in Figure 5.9, the decision tree is three levels deep. The size of the decision-tree is governed by two factors:

(i) The number of actual states s visited while following the close-to-optimal policy. Let us say $S_v \in S$ is the set of actual states visited and $S_v' = S / S_v$. The size of the decision-tree depends on the size of S_v. When the close-to-optimal policy is implemented, the belief dimensions corresponding to S_v' contribute less and less to decision- making.

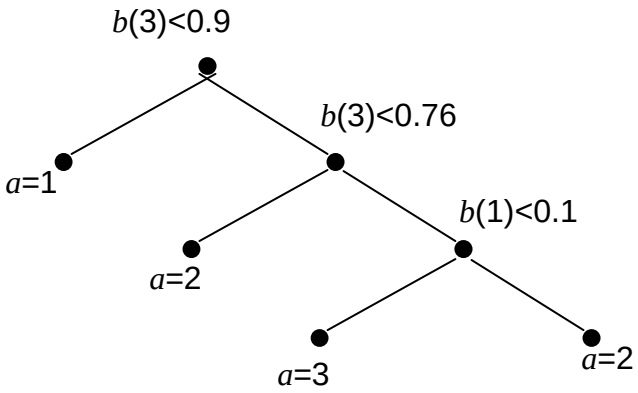


Figure 5.9: Decision tree for the single machine problem. Values of 1,2 and 3 for *a* represent the decisions to 'do nothing' , 'inspect the job' and 'service the equipment' respectively.

(ii) The level of similarity between actual states *s* – due to the similarity, a cluster of states would correspond to the same (near)optimal action. In the region of belief states, the states belonging to these clusters would form hyper-planes and lead to a decision-tree of much lower dimension. The sizes of the decision-trees are also reported in Table 5.2 and are substantially smaller than the size of S_v. This indicates the formation of clusters of states that behave in a similar manner.

5.6 **Conclusions**

Judging by the research efforts in the area of partially observed degradation of manufacturing equipment and costly inspection, the extension of the concepts to multiple operations is important. In this work this problem is addressed for a re-entrant and a hybrid flow topology. The significance of rigorous treatment of this class of problems is illustrated by comparing the results with those of heuristic methods. The POMDP formulations result in significant improvements in performance over those of best heuristics. However, the POMDP problem grows fast with increasing problem sizes. Therefore, characterization of the (near) optimal policies is an important direction for future work in this area.

CHAPTER 6

MILP BASED VALUE BACKUPS IN POMDPS WITH VERY LARGE OR CONTINUOUS ACTION SPACES

6.1 Introduction

In the last chapter we saw that POMDPs serve as powerful tools to model stochastic systems with partial state information. Since the exact solution methods for POMDPs are limited to problems with very small sizes of state, action and observation spaces, approximate solution methods have gained popularity. Notable among these are the point based methods which consider a fixed or evolving set of prototype belief points instead of considering the entire belief simplex. A particular point based method, PERSEUS (Spaan and Vlassis, 2005) was used in chapter owing to the fact that it favorably makes use of the piecewise linear and convex (PWLC) structure of the value function to speed up convergence. In this chapter, we consider POMDPs with very large or continuous action space. In the current form of PERSEUS and many other point based methods, presence of continuous actions or very large action space makes it practically impossible to compute the value backup exactly. We use a mathematical program to circumvent this difficulty. In an alternative formulation, we develop value iteration update equations around post decision belief state as opposed to the traditionally used notion of (pre-decision) belief state. Depending on the size of the observation space, latter approach reduces the computational load of the mathematical program based value backup calculation. The requirements on the structure of the reward function and the dependence of probabilities of state transition and observation on action are provided. Two illustrations, one each for pre-decision and post-decision belief states are included to analyze the efficacy of the method. Comparison with enumeration and action sampling methods is provided in terms of solution quality and solution time.

6.2 Related work

Adopting the POMDP notation from Chapters 2 and 5, the value backup for a belief point b for (infinite horizon POMDP with discount factor γ) is given by (6.1), where $\alpha_n^i \; i = 1,2,.. \mid V_n \mid$ is the set of gradient vectors that characterizes the value function at n^{th} iteration. We use infinite horizon POMDP with discounting in all illustrations. Equivalent models can be derived for finite horizon POMDPs with little difficulty.

$$V_{n+1}(b) = \max_{a \in A} \{ \sum_{s \in S} r(s,a) b(s) + \gamma \sum_{o \in O} p(o \mid b, a) V_n(b^{a,o}) \} \tag{6.1}$$

$$b^{a,o}(s') = \frac{\sum_{s \in S} p(s' \mid s, a)\, p(o \mid s', a) b(s)}{\sum_{s' \in S} \sum_{s \in S} p(s' \mid s, a)\, p(o \mid s', a) b(s)} \tag{6.2}$$

$$p(o \mid b, a) = \sum_{s' \in S} \sum_{s \in S} p(s' \mid s, a)\, p(o \mid s', a) b(s) \tag{6.3}$$

$$V_n(b^{a,o}) = \max_i \sum_{s' \in S} \alpha_n^i(s') b^{a,o}(s') \tag{6.4}$$

As with fully observable Markov decision processes (FO-MDP or simply MDP), the presence of large and continuous action spaces poses a challenge in terms of computation of maximization over the entire action space (6.1). Additionally, in case of the exact solution methods for POMDPs, the size of action space, i.e. $|A|$ also affects the number of potential gradient vectors which may comprise the value function at the next iteration. The two aspects are summarized below:

(i) In each value iteration step and in the policy evaluation step of policy iteration algorithm, the maximization is performed over the entire action space for all or a subset of prototype belief points. Therefore, for every iteration, the computation time is proportional to $|A|$ when enumeration of all actions is used.

(ii) For exact solution methods reported by Monahan (1982); Hauskretch (2000), in order to obtain the value function estimate during n^{th} iteration, a set of all possible gradient

vectors is generated. Linear Programming (LP) is then used to determine the useful among these vectors. The term *useful* denotes the gradient vector that maximizes the value of at least one point in the belief simplex as compared with all other gradient vectors. The size of all possible gradient vectors at the n^{th} iteration is $|V_n||A||O|$, where $|V_n|$ is the number of gradient vectors that characterize V_n.

It is not surprising then that, to the best of our knowledge, no current solution method claims to compute the max operation exactly for very large or continuous action spaces. A subset of literature that considers POMDPs with very large or continuous action spaces is reviewed below.

Policy search methods can handle continuous action spaces. An example is Pegasus (Ng and Jordan, 2000), which estimates the value of a policy by simulating trajectories from the POMDP using a fixed random seed, and adapts its policy in order to maximize this value. Pegasus can handle continuous action spaces at the cost of a sample complexity that is polynomial in the size of the state space. Baxter and Bartlett (2001) propose a policy gradient method that searches in the space of randomized policies, and which can also handle continuous actions. The main disadvantages of policy search methods are the need to choose a particular policy class and the fact that they are prone to local optima. Thrun (2000) and Spaan and Vlassis, (2005) consider sampling techniques to keep the active size of the action space relatively small for continuous or very large action spaces. The method is problem dependent and may lead to loss of solution quality in certain applications.

In the Monte Carlo POMDP (MC-POMDP) method of Thrun (2000), real-time dynamic programming is applied on a POMDP with a continuous state and action space. In that work, beliefs are represented by sets of samples drawn from the state space, while $Q(b,a)$ values are approximated by nearest-neighbor interpolation from a (growing) set of prototype values and are updated by online exploration and the use of sampling-based Bellman backups. In contrast with PERSEUS, the MC-POMDP method does not exploit the piecewise linearity and convexity of the value function.

6.3 Mathematical programming based value updates

6.3.1 Formulation of the mathematical program

The biggest motivation for using a mathematical program to compute the value backup for a belief point *b* is the fact that the value function for infinite horizon POMDP can be approximated well by a PWLC function (Sondik 1978). The value backup equation is shown in (6.1). Assuming that the state and observation spaces are finite and with a little abuse of notation, the reward function $r_a(s) = r(s,a)$, state transition probability function $t_a(s,s') = T(a,s,s')$ and observation probability function $op_a(s',o) = OP(s',o)$ are dependent on action *a* as shown in (6.5) through (6.7). Here subscript *a* suggests dependence on action *a*; *s*,*s′* and *o* represent the indices of current state, next state and observation respectively.

$$r_a(s) = f_1(a) \qquad \forall s \tag{6.5}$$

$$t_a(s,s') = f_2(a) \qquad \forall s,s' \tag{6.6}$$

$$op_a(s',o) = f_3(a) \qquad \forall s',o \tag{6.7}$$

The equivalent mathematical program for (6.1) can be written as shown in (6.8) through (6.13). $\alpha_n(i,s') = \alpha_n^i(s')$ is used for ease of notation.

$$\max_{a\in A}\{\sum_{s\in S} r_a(s)b(s) + \gamma\sum_{o\in O}\sum_{s'\in S}\sum_{s\in S} t_a(s,s')op_a(s',o)b(s)v(o)\} \tag{6.8}$$

s.t.

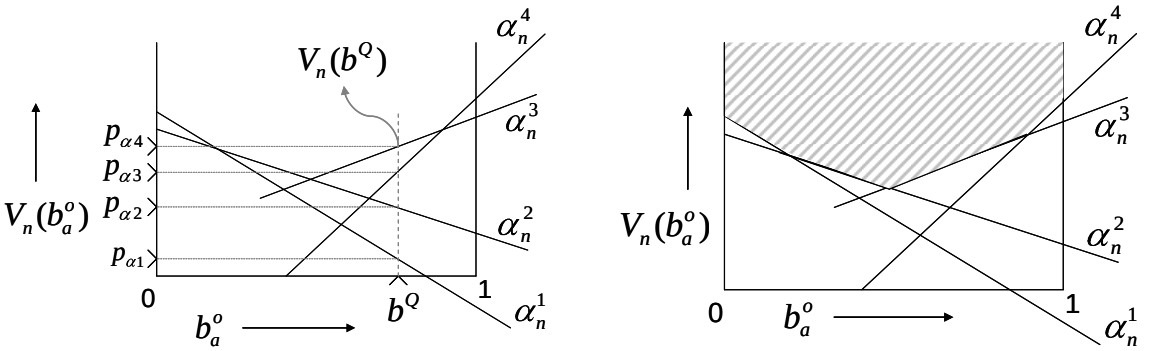
$$v(o) = \max_i \sum_{s'\in S} \alpha_n(i,s')b^{a,o}(s') \quad \forall o \tag{6.9}$$

$$b^{a,o}(s') = \frac{\sum_{s\in S} t_a(s,s')op_a(s',o)b(s)}{\sum_{s'\in S}\sum_{s\in S} t_a(s,s')op_a(s',o)b(s)} \qquad \forall s' \tag{6.10}$$

$$r_a(s) = f_1(a) \qquad \forall s \tag{6.11}$$

$$t_a(s,s') = f_2(a) \qquad \forall s,s' \tag{6.12}$$

$$op_a(s',o) = f_3(a) \qquad \forall s',o \tag{6.13}$$



(a) (b)

Figure 6.1: (a) Value function calculation at a query point (b) Feasible region for the MILP model for value update

Figure 6.1(a) represents (6.9) for a query point b^Q for a two state problem. The system can be in one of two states (s_1 and s_2) at any time. The *x*-axis represents the probability of being in state s_2, *y*-axis shows the value function estimate during n^{th} iteration. This is represented by means of gradient vectors α_n^i. For a query point $b^{a,o} = b^Q$, (6.8) takes the maximum of points $p_{\alpha i}$ *i*=1,2,..|V_n|. In order to accomplish this, a set of constraints (6.14) is used to define the feasible region shown by the shaded region in Figure 6.1(b).

$$v(o) \geq \sum_{s' \in S} \alpha_n(i,s') b^{a,o}(s') \quad \forall i,o \tag{6.14}$$

However, $\sum_o v(o)$ is maximized in the objective function. This would make the problem unbounded. In order to set $v(o)$ equal to $\max_i \sum_{s' \in S} \alpha_n(i,s') b^{a,o}(s')$, additional sets of constraints (6.15) and (6.16) are introduced by using a set of binary variables $y(i,o)$. M is chosen such that constraint (6.14) is not violated.

$$v(o) \leq \sum_{s' \in S} \alpha_n(i,s') b^{a,o}(s') + M(1 - y(i,o)) \quad \forall i,o \tag{6.15}$$

$$\sum_i y(i,o) = 1 \qquad \forall o, y(i,o) \in \{0,1\} \tag{6.16}$$

This requires that (6.17) be satisfied. In a general case, this is achieved by setting $M = \max_i \{\max_{s'} \alpha_n(i,s')\}$. Therefore, constraint sets (6.14) through (6.16) exactly model the equations (6.9), when requirement (6.17) is met.

$$M \geq \max_i \sum_{s' \in S} \alpha_n(i,s') b^{a,o}(s') - \sum_{s' \in S} \alpha_n(i,s') b^{a,o}(s') \quad \forall i,o \tag{6.17}$$

Substituting the value of $b^{a,o}$ from (6.10) and canceling the term $\sum_{s \in S} \sum_{s' \in S} t_a(s,s') op_a(s',o) b(s)$, the resultant mathematical program is shown in Figure 6.2. Admittedly, the reward and probability functions in the form of f_1,f_2 and f_3 have not been defined. These functions are better understood by illustrative examples presented in section 6.5.

$$\max_{a \in A} \{ \sum_{s \in S} r_a(s) b(s) + \gamma \sum_{o \in O} v(o) \} \quad \text{M1.1}$$

s.t.

$$v(o) \geq \sum_{s' \in S} \sum_{s \in S} t_a(s,s') op_a(s',o) \alpha_n(i,s') b(s) \quad \forall i,o \quad \text{M1.2}$$

$$v(o) \leq \sum_{s' \in S} \sum_{s \in S} t_a(s,s') op_a(s',o) \alpha_n(i,s') b(s) + M(1 - y(i,o)) \quad \forall i,o \quad \text{M1.3}$$

$$\sum_i y(i,o) = 1 \quad \forall o \quad \text{M1.4}$$

$$r_a(s) = f_1(a) \quad \forall s \quad \text{M1.5}$$

$$t_a(s,s') = f_2(a) \quad \forall s,s' \quad \text{M1.6}$$

$$op_a(s',o) = f_3(a) \quad \forall s',o \quad \text{M1.7}$$

Figure 6.2: The MILP for determination of the maximizing action for value update

6.3.2 Computational efficiency of the mixed integer formulation

The value backups are computed many times for different belief states in each iteration. The operation is then repeated for multiple iterations. It is therefore imperative that the mathematical program associated with the value backup be computationally efficient and yield near-optimal solutions for each solve. In order to ensure the above two properties, restrictions on the structure of the mathematical program need to be imposed. This limits the applicability of the proposed approach to a certain extent. In general, a linear, quadratic or convex program provides ease of computation. This requires that the stage-wise reward, equations and constraints be a linear, quadratic or convex function of the action. Due to the presence of integer variables, a linear formulation is most suitable. We present model requirements for the mathematical program to be linear. Similar analysis may be carried out for quadratic or convex programs.

We first list the requirements on reward and probability functions for the model to be linear and then provide ways to work around some of the requirements. It must be noted that linearity of the mathematical program is required to assure (near) optimality of the value backups and to keep the computational load to reasonable limits.

It is easy to see that linearity of the model is determined by the structure of functions f_1,f_2 and f_3. Due to equations M1.5 through M1.7, the following restrictions are placed:

(i) f_1 is a linear function of action a

(ii) $f_2 f_3$ is a linear function of a

Condition (ii) essentially implies that at least one of the functions (f_2 and f_3) may not depend on a. This is imposed by constraints M1.2 and M1.3.

Additionally, when functions f_1,f_2 or f_3 involve use of binary variables to represent logical constraints, the linearity of the model may be affected. Finally, when the system state has multiple dimensions, it may be necessary to include the entire state description as opposed to the state indices s and s' in order to determine stage-wise reward and probabilities. This aspect is covered in illustration 2.

Whereas condition (i) is true of many real world systems, at the first glance, condition (ii) appears to be overly restrictive to warrant successful application of the method. However, a closer look reveals that many real world systems may be modeled while satisfying condition (ii). It is generally seen that state transition probabilities depend strongly on action choice, i.e. f_2 is a function of a. Observation probabilities (if at all) depend on discrete actions like whether a sensor measurement is taken. Alternatively, the discrete decisions that affect observation probabilities may include which or how many sensors are used for measurement/taking the observation. These discrete decisions can be made a part of state description leading to observation probabilities depending only on state s' and observation o, but independent of action a.

6.3.3 Implementation and policy determination

POMDP solution algorithm

The mathematical program shown in Figure 6.2 is a general model to determine value backup for a belief point b when the parameterized form of value function is used. PERSEUS, described in Chapter 2 and used for solving the POMDPs in Chapter 5 is one such algorithm. It directly uses the parametric form of the value function by maintaining a finite set of gradient vectors α_n. Gradient updates are obtained together with value updates around a prototype belief set, B, as outlined in section 2.3 of Chapter 2. The updated vector $\alpha_{n+1}^{\{b\}}$ is admitted in V_{n+1} only if it improves the value at point b as compared with V_n. Otherwise, $\alpha_n^{\{b\}}$ is admitted. When value backups are obtained using MILP, the gradient can be easily calculated using the maximizing action $a^{*\{b\}}$. It is also possible to obtain the gradient vector directly from the MILP solve (but this is not considered here).

Policy determination

Having obtained the estimate of optimal value function that satisfies a desired convergence criterion ε, the ε-optimal policy can be obtained using one of the following methods

(i) Maximizing action associated with each gradient vector – During a value backup for belief point b, a gradient vector $\alpha_{n+1}^{\{b\}}$ is obtained which has a maximizing action $a^{*\{b\}}$ (6.18) associated with it. The corresponding maximizing vectors can be cached with each gradient vector. The ε-optimal policy $\pi^{\varepsilon}(b)$ for belief point b is then given by (6.19)

$$a^{*\{b\}} = \arg\max_i \sum_s b(s)\alpha^i_{|V^{\varepsilon}|}(s) \tag{6.18}$$

$$\pi^{\varepsilon}(b) = a^{*\{b\}} \tag{6.19}$$

However, it may be expensive to hold the entire action descriptions, for each gradient vector, when the dimension of actions is large.

(ii) One step look ahead controller – The ε-optimal policy $\pi^{\varepsilon LA}(b)$ for a belief point b can also be obtained in real time by running the value backup operation at each time. This is shown in (6.20), together with (6.2) , (6.3) and (6.4).

$$\pi^{\in LA}(b) = \arg\max_{a \in A} \{\sum_{s \in S} r_a(s) b(s) + \gamma \sum_{o \in O} p(o \mid b, a) V^{\varepsilon}(b^{a,o})\} \tag{6.20}$$

Generally speaking, the first approach is computationally more favorable but may have high memory requirements if the dimension of the actions is large. It can be shown that policy $\pi^{\in LA}$ is at least as good as $\pi^{\in}$ in an average sense.

6.3.4 Problem size v/s computational complexity

As seen from the model in Figure 6.2, the size of the model (number of variables and constraints) is directly proportional to $|S|^2$, $|O|$ and $|V_n|$. Dependence of $|A|$ is implicit. Since f_1, f_2 and f_3 are transition functions, in the absence of logical variables they pose little computational challenge. The size of the model is greatly affected by the number of integer variables i.e., $y(i,o)$ in this case. This number clearly depends on $|V_n|$ and $|O|$. While $|O|$ comes directly from the model, the size of V_n is governed by a combination of factors. The most important factor is the dimensionality of belief simplex. In terms of PERSEUS, a higher dimensional simplex would require higher number of belief points comprising the prototype belief set and $|V_n| \leq |B|$. However, in practice $|V_n| << |B|$ (Spaan and Vlassis, 2005). Another determinant of $|V_n|$ is the structure of optimal policy. E.g., when the decision region is convex, it is possible to approximate the function corresponding to each decision region by very few gradient vectors. In this case, $|V_n|$ would depend on $|A|$. In general however, it is difficult to predict the size of V_n from the model. When $|V_n|$ is very large, we propose some heuristics to pick from the existing (potentially large) set of gradient vectors, the set which has the most potential for representing the entire value function. The idea is to construct an active set of gradient

vectors V_{n_active} to feed into the MILP during each iteration. The size of the active set cannot exceed a pre-defined limit δ, which is chosen to keep the computation time for each backup within prescribed limits.

(i) Top pick approach – In this approach, the gradient vectors belonging to V_n are arranged in decreasing order of the total number of belief points that they maximize. When $|V_n|>\delta$, the top fraction $\eta\delta$ are made a part of the action set V_{n_active}. In order to avoid local optima, the remaining fraction $(1-\eta)\delta$ is chosen randomly from V_n. The parameter η represents the classic trade-off between exploration and exploitation during optimization. Starting with a value close to 0.5, η may be increased gradually with the iteration count n.

(ii) Nearest neighbor approach – This approach is based on the fact that certain gradient vectors may only marginally improve the value at a belief point over some other vector. For each belief state, it is possible to obtain the gradient that maximizes the value at the point and also the gradient that gives the next best value. From the set of maximizing gradient and the next best gradient for each belief point, it is possible to obtain a set of gradient vectors whose maximum size $|B|/2$.

The gradient sampling technique is similar in spirit with action sampling methods. While the size of action space grows exponentially with the dimension of action, $|V_n|$ generally grows exponentially with $|S|$. Efficiency of either methods (enumeration or MILP based value backups) would be determined by the particular application.

While the above approaches to limit the size of V_n appear rather brute force, there is a more elegant way to resolve the problem of large sized observation spaces. This is addressed in the following section.

6.4 Value iteration around post decision belief state

6.4.1 The basic idea

For a general MDP (fully observable), the notion of post decision state applies to problems where the effect of actions and uncertainty on state variable can be separately

represented. Since POMDP is equivalent to a continuous state FO-MDP, this concept can be utilized here, given the aforementioned requirement is met. E.g., at time *t*, let the belief state be denoted by b_t. When action a_t is taken, the state transitions to an intermediate state $\tilde{b}_t^a$ while uncertainty is not yet realized. Finally, uncertainty in the form of observation *o* is realized and the system can be in any of |*O*| next states where |*O*| is the size of uncertainty, i.e., the number of possible next beliefs. This is schematically shown in Figure 6.3. Circles represent the more popular pre-decision belief state b_t, b_{t+1} etc. and squares represent the intermediate state $\tilde{b}_t^a$ that captures the effect of action only. This is referred to post decision belief state.

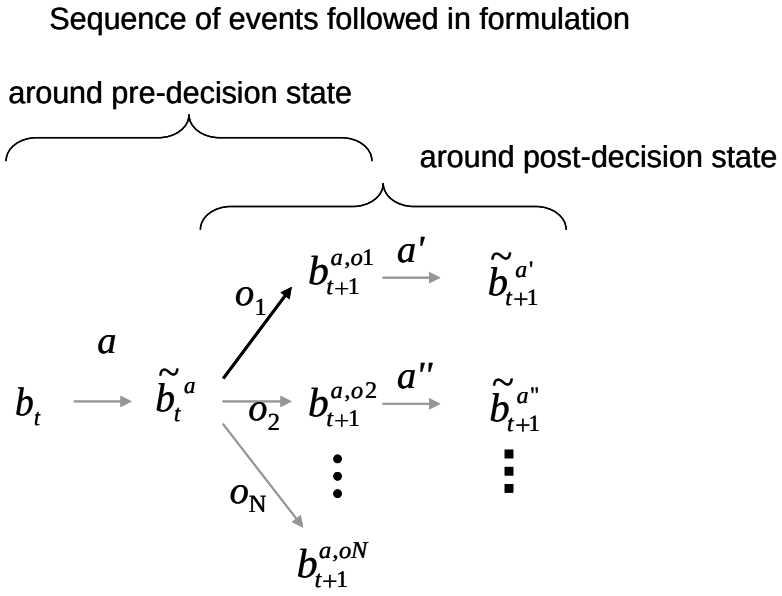


Figure 6.3: A schematic of pre-decision state to post-decision state and again to pre-decision state

In the context of POMDPs, solution using this approach is possible when the observation probabilities do not depend on action a. As pointed out in section 6.3.2, the actions that affect observation probabilities may be made part of the state. The transition from regular belief state b_t to $\tilde{b}_t^a$ then is simply given by $t_a b_t$.

It is to be noted that although the effect of action on underlying states $s \in S$ may be prone to uncertainty, the belief state transition ($t_a b_t$) is *always* deterministic. Having obtained the post decision belief state, $|O|$ (pre-decision) belief states may be obtained at the next time step. The transition is governed by the observation probabilities $p(o|b_t)=p(o|\tilde{b}_t^a)$ for all $o \in O$. This two step transition is shown in Figure 6.3. Intuitively, starting with a post decision state $\tilde{b}^a$, $|O|$ possible next states $b^{a,o}\ \forall o$ are obtained. For each of the states $b^{a,o}\ \forall o$, a maximizing action $a'^{*\{o\}}$ would determine the next post decision belief state $b^{a',o}$. Consequently, the value iteration update takes the form shown in (6.22) through (6.26). V^a represents value function around post-decision belief state $\tilde{b}^a$. For notational ease, the dependence of action a' and next post-decision belief state $\tilde{b}^{a'}$ on o is suppressed in future illustrations. The details on the derivation of value iteration equation around post-decision state can be found in (Powell 2007) for a general MDP.

In (6.22), it is to be noted that the expectation over observations is outside the max operator. This removes the dependence of the size of the MILP on $|O|$. However, it still remains to be shown that the structure of V^a is PWLC. To this end we first show that

$$V(\tilde{b}^a) = \sum_o p(o \mid \tilde{b}^a, a) V^{pre}(b^{ao}) \tag{6.21}$$

and then use Lemma 1 to show that a positive weighted sum of convex functions is convex. Since $p(o \mid \tilde{b}^a) \geq 0$, the result holds.

Lemma 1*:*

For two convex functions f and f' and scalar $\alpha \geq 0$
i) αf is convex
ii) $f + f'$ is convex

In the following sections, we derive the value and gradient update equations around post-decision belief state and then present pros and cons of using this method over regular value iteration around pre-decision state.

6.4.2 Derivation of backup equations and formulation of math program

The equations for value and gradient backups around post decision belief states are deriveed on similar lines as shown in (Spaan and Vlassis, 2005). As seen in the previous section, the value iteration step for the post decision state variable is given by (6.22) through (6.26).

$$V_{n+1}^{a}(\tilde{b}^{a}) = \sum_{o} p(o \mid \tilde{b}^{a}, a) \max_{a'} \{ \sum_{s} r(s,a') b^{ao}(s) + \gamma V_{n}^{a}(\tilde{b}^{a'}(s')) \} \tag{6.22}$$

where

$$b^{ao}(s) = \frac{\tilde{b}^{a}(s)\, p(o \mid s,a)}{\sum_{s} \tilde{b}^{a}(s)\, p(o \mid s,a)} \tag{6.23}$$

$$\tilde{b}^{a'}(s') = \sum_{s} b^{ao}(s)\, p(s' \mid s,a') \tag{6.24}$$

$$V_{n}^{a}(\tilde{b}^{a'}(s')) = \max_{i} \sum_{s'} \tilde{b}^{a'}(s') \alpha_{n}^{i}(s') \tag{6.25}$$

$$p(o \mid \tilde{b}^{a}, a) = \sum_{s} p(o \mid s,a) \tilde{b}^{a}(s) \tag{6.26}$$

Substituting (6.23) through (6.26) in (6.22)

$$V_n^a(\tilde{b}^a) = \sum_o p(o \mid \tilde{b}^a, a) \max_{a'} \{ \frac{\sum_s r(s,a') p(o \mid s,a) \tilde{b}^a(s)}{\sum_s p(o \mid s,a) \tilde{b}^a(s)} + \gamma \max_i \sum_{s'} \sum_s \frac{p(o \mid s,a) p(s' \mid s,a') \tilde{b}^a(s) \alpha_n^i(s')}{\sum_s p(o \mid s,a) \tilde{b}^a(s)} \} \quad (6.27)$$

Since $p(o \mid \tilde{b}^a, a)$ is independent of a', it is taken out of max and cancelled with the numerator. This implies that

$$V_{n+1}^a(\tilde{b}^a) = \sum_o \max_{a'} \{ \sum_s r(s,a') p(o \mid s,a) \tilde{b}^a(s) + \gamma \max_i \sum_{s'} \sum_s p(o \mid s,a) p(s' \mid s,a') \tilde{b}^a(s) \alpha_n^i(s') \} \quad (6.28)$$

using

$$g_{a'o}^i(s) = \sum_{s'} p(o \mid s,a) p(s' \mid s,a') \alpha_n^i(s') \quad (6.29)$$

$$and \;\; T^{a'o}(s) = r(s,a') p(o \mid s,a) \quad (6.30)$$

and the identity

$$\max_{\{y_j\}_j} x.y_j = x.\arg\max_{\{y_j\}_j} x.y_j \quad (6.31)$$

$$V_{n+1}^a(\tilde{b}^a) = \sum_o \max_{a'} \{ \sum_s T^{a'o}(s) \tilde{b}^a(s) + \gamma \tilde{b}^a.\arg\max_{\{g_{a'o}^i\}_i} < \tilde{b}^a, g_{a'o}^i > \} \quad (6.32)$$

$$G^{a'o}(s) = T^{a'o}(s) + \gamma \arg\max_{\{g_{a'o}^i\}_i} < \tilde{b}^a, g_{a'o}^i > \quad (6.33)$$

$$V_{n+1}^a(\tilde{b}^a) = \sum_o \{ \tilde{b}^a.\arg\max_{\{G^{a'o}\}_{a'}} < \tilde{b}^a, G^{a'o} > \} \quad (6.34)$$

and

$$\alpha_{n+1}^{\{\tilde{b}^a\}} = \sum_o \arg\max_{\{G^{a'o}\}_{a'}} < \tilde{b}^a, G^{a'o} > \quad (6.35)$$

(6.34) and (6.35) give the value backup and gradient vector backup for the post decision belief state variable respectively.

On lines similar to section 6.2.1, the mathematical program for (6.32) for a given observation and belief state $\tilde{b}^a$ is shown in Figure 6.4. Evidently, the binary variables $y(i)$ for i=1,2,. $|V_n|$ and variable v do not depend on observation o. However the model has to be solved multiple times to obtain the backup. The number of times the model is to be solved is given by $o\_size \leq |O|$, where o_size is the number of observations for which $p(o|\tilde{b}^a)>0$.

$$\max_{a\in A}\{\sum_{s\in S} r_{a'}(s)b^{a,o}(s) + \gamma v(o)\}$$

$$s.t.$$

$$v(o) \geq \sum_{s'\in S}\sum_{s\in S} t_a(s,s')\alpha_n(i,s')b(s) \quad \forall i$$

$$v(o) \leq \sum_{s'\in S}\sum_{s\in S} t_a(s,s')\alpha_n(i,s')b(s) + M(1-y(i)) \quad \forall i$$

$$\sum_i y(i) = 1$$

$$r_a(s) = f_1(a) \quad \forall s$$

$$t_a(s,s') = f_2(a) \quad \forall s$$

Figure 6.4: The MILP for determination of the maximizing action for value update for a post-decision belief state

Policy determination

Similar to the pre-decision state case of section 6.3.2, there are two possibilities to determine the optimal action for a belief state b, i.e., (i) Storing the maximizing action(s) associated with each gradient vector and (ii) One step look-ahead controller. However, there are multiple actions associated with a gradient backup ($a'^{*\{o\}} \ \forall o$). Therefore the

action associated with each observation needs to be cached. This results in higher memory requirement for storing the ε-optimal policy for the post-decision state, as compared to that for the pre-decision state. For the look-ahead design on the other hand, the MILP needs to be solved for the belief state pertaining to the current observation only. This is computationally less expensive than the look-ahead design for solution around pre-decision belief state.

6.4.3 Comparison with value updates around pre-decision belief states

While using MILP based value updates the two methods can be compared along following avenues:

i) Complexity of MILP problem – While using the post formulation several smaller MILPs are solved for one value backup as opposed to solving one large MILP for pre-formulation. The former almost always works better if the input/output operations between the MILP solver (e.g. CPLEX) and regular solution platform (e.g. MATLAB) are not as time consuming as the optimization itself.

ii) Policy determination in real time – As discussed in the previous section and section 6.3.3, the formulation around pre-decision state allows for storing optimal action with each gradient vector that characterizes the ε-optimal value function. For formulation around post-decision belief state optimal action needs to be stored for each gradient vector and each observation. This increases the memory requirement for the latter. This may not be feasible in the post formulation when the observation space is very high. However, the look-ahead design for policy determination is faster for the post-formulation due to lower complexity of the associated MILP. Consequently, policy determination in real time in post-formulation can be prohibitively slower.

iii) Handling very large or continuous observation spaces – While pre-formulation is limited to small observation spaces, the MILP technique for value updates based on post-formulation is on par with enumeration based methods. When observation space is very large or continuous, Hoey and Poupart (2005) consider creating sets of

observations for which b leads to the same future belief state $b^{a,o}$. This effectively makes the observation space discrete. Such manipulation of observation probabilities is better achieved outside the confines of a mathematical program.

Aside from (ii) above, it is easy to see that the two formulations are similar in terms of computational complexity when enumeration of action space is used for value backups. This consideration excludes the fact that post formulation allows for parallel processing of max operation. Having derived the necessary models and equations, the following section is devoted to two illustrative examples. The examples are aimed at demonstration of technique and analysis of solution times and solution quality as a function of problem size.

6.5 Illustrative examples

The first example in this section contains continuous actions and the POMDP is formulated around pre-decision belief state. For simplicity, the observation probabilities are assumed independent of action in both illustrations.

6.5.1 POMDP with continuous actions

In order to illustrate the concept of using mathematical programming for value backups, a simple problem with two states is considered first. A hypothetical equipment can be in one of two states (s_1 and s_2) at any time. The system probabilistically transitions between the two states.

Rewards R_1 and R_2 are received when system is in state s_1 and s_2 respectively and $R_1 > R_2$. This implies that s_1 is more desirable state than state s_2. There are two possible actions a_1 and a_2 which affect the probabilities of state transition as shown below. $a_1 \in (0,1)$ and $a_2 \in (0,1)$ are continuous and bounded. While a higher value of a_1 helps the system remain in state 1, a higher value of a_2 ensures it's returning to state s_1 from state s_2. These actions can be thought of as routine preventive and corrective actions to make sure the equipment is in state s_1, e.g. cleaning, lubrication etc. The tasks are scaled to obtain the bounds of 0 and 1.

$$S = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad A = \begin{bmatrix} a1 \\ a2 \end{bmatrix} \quad O = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad T_a = \begin{bmatrix} 0.5(1+a_1) & 0.5(1-a_1) \\ a_2 & 1-a_2 \end{bmatrix}$$

The (state dependent) unit costs of taking action a_1 and a_2 in state s_1 are C_{11} and C_{12} respectively. Similarly, the unit costs of taking action a_1 and a_2 in state s_2 are C_{21} and C_{22} respectively with $C_{12}>C_{11}$ and $C_{22}>C_{21}$ .This ensures that the cost of keeping the system in state s_1 is lower than bringing it back to s_1, when it has transitioned to s_2. Accurate state observation is made with probability $0<\beta<1$. The resulting observation and reward matrices are as shown below:

$$OP = \begin{bmatrix} \beta & 1-\beta \\ 1-\beta & \beta \end{bmatrix} \qquad R_a = \begin{bmatrix} R_1 - C_{11}a_1 - C_{12}a_2 \\ R_2 - C_{21}a_1 - C_{22}a_2 \end{bmatrix}$$

Finally, there are system and budget constraints of the form shown in (6.36) and (6.37) respectively. While the former specify system requirements, e.g., a certain mix of cleaning fluids from the two actions, the latter represent limits on total expenditure. Since cost of actions is state dependent, the probabilities of being in state s_1 and s_2, i.e., $b(s_1)=b_1$ and $b(s_2)=b_2$, are also a part of the constraints.

The feasible region of the MILP, in the absence of constraints (6.36, 6.37), is given by the shaded region with hashed line in Figure 6.5. Therefore, in the absence of constraints (6.36, 6.37), the solution of the above problem would lie on the extreme points, i.e., optimal values of both a_1 and a_2 are either 0 or 1. In such a scenario, the action space is practically discrete and substantially smaller.

$$A_1a_1 + A_2a_2 + A_3 \leq 0 \tag{6.36}$$

$$B_{11}b_1a_1 + B_{12}b_1a_2 + B_{21}b_2a_1 + B_{22}b_2a_2 + B_3 \leq 0 \tag{6.37}$$

The parameter values considered for this study are shown in Table 6.1.

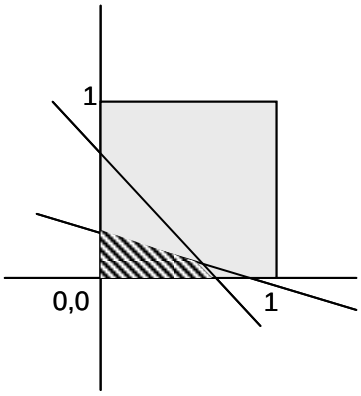


Figure 6.5: Feasible region for the continuous two-action problem

Model M2

$$\max_a \{\sum_s b^o(s) r(s) + \gamma \sum_o v(o)\}$$

s.t.

$$r(s) = R(s) - C(s,s')a(s')$$

$$t = Aa + B$$

$$v(o) \geq b_o tp(o)\alpha$$

Table 6.1: Parameter values for the system with two-continuous actions

Parameter	β	R_1	R_2	C_{11}	C_{12}	C_{21}	C_{22}	A_1
Value	0.9	2	0	0.1	0.15	0.3	0.05	0.5088
Parameter	A_2	A_3	B_{11}	B_{12}	B_{21}	B_{22}	B_3	A_2
Value	1.325	-1	3.1185	9.355	3.0736	1.025	-1	1.325

The MILP model for the value backups for above example is shown by model M2. This pertains to value iteration around pre-decision state variable.

Optimal policy and value function

The converged value function and policy for both approaches are shown in Figure 6.6. When the probability of being in state s=1 is sufficiently high, action a_1 is executed whose value depends on the constraints and the objective value. The policy for the two action problem is rather simple and intuitive. When the probability of being in state s=1 falls below a certain threshold (different for both approaches), action a_2 is performed so that it satisfies the constraints listed in (6.36) and (6.37). As seen in Figure 6.6, the value function for the enumeration based solution is lower. The performance and solution times of MILP and enumeration based methods (as a function of number of PERSEUS iterations) are shown in Figures 6.7(a) and 6.7(d) respectively. The enumeration based method convergences in an order of magnitude less time than the MILP based method. However, the performance of the enumeration based method is lower than that of the MILP solution. This is attributed to the discretization of the action space. The action corresponding to each gradient vector is also shown in Figure 6.6. As seen, the actions pertaining to the best solution obtained by enumeration of the value function is limited by the size of the action space grid. For the same reason, the MILP based method appears to be converging faster in terms of number of iterations as seen in Figure 6.7(d).

Scalability

To study how the solution time scales with the problem size and understand whether the value gap depends on the problem size, two additional experiments are performed: (i) a system with three possible states and three actions, (ii) a system with four states and four actions. Similar to the two-action system above, the states are discrete and all actions are continuous and range from 0 to 1. The probability transition matrix is a linear function of actions and the constraints follow the same linear structure as (6.36) and (6.37). The parameter values are shown in Table 6.1 and the results are reported in Table 6.2. It is observed that the performance gap widens as the problem size grows. The

phenomenon is seen in Figures 6.7(d) through 6.7(f) as well. Additionally, unlike the case with the two-action system, the convergence times are of the same order of magnitude for three-action system and the convergence time is an order of magnitude higher for the enumeration based technique as the problem size grows to four actions. As the problem size grows, the number of actions grows exponentially in the case of the enumeration based method. Also due to coarse-grid sampling of the action space, the technique tends to maintain many more gradient vectors to closely approximate the value function. The two factors, i.e. $|A|$ and $|V_n|$, contribute to the rapid increase in solution times as the problem size increases. For the MILP based method, the number of actions as well as the number of gradient vectors increase moderately with problem size. The solution times therefore grow marginally as the problem size grows. This can be concluded from the fact that the plots corresponding to MILP based method are almost identical.

The shape of the solution time v/s number of iterations plot, reveals information about the dependence of solution time per iteration on $|V_n|$ where n is the iteration counter. A discussion on this dependence is deferred until the network flow example presented in the next section.

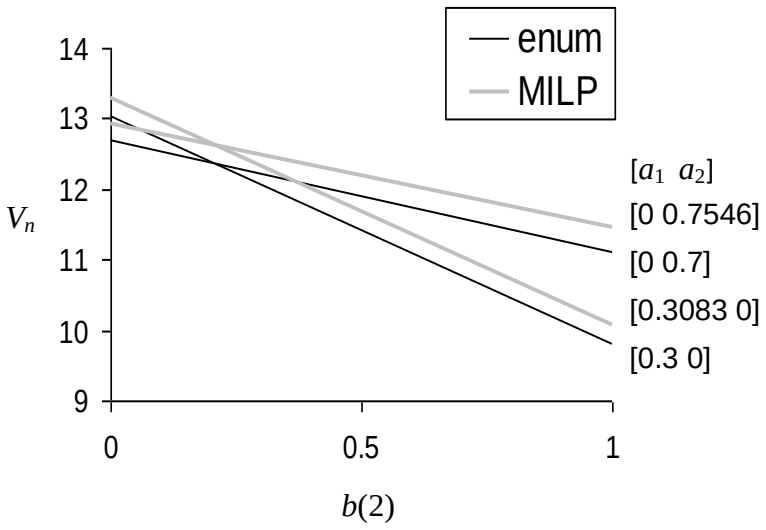


Figure 6.6. Value function and policy comparison for two action system

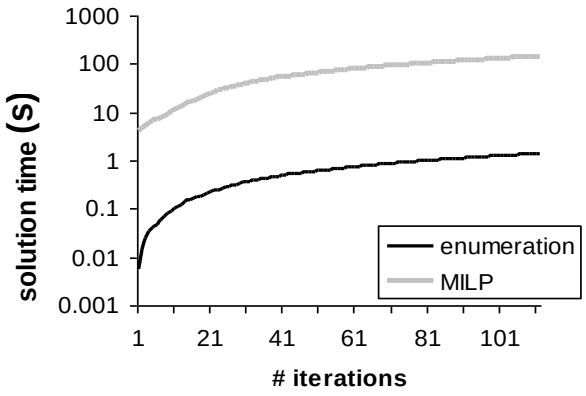
Table 6.2. Results for the problems with continuous actions

Problem	$\|S\|$	$\|A\|$	$\|O\|$	performance		$\|V_n\|$		convergence time	
				MILP	enum	MILP	Enum	MILP	Enum
2 actions	2	121	2	13.28 ±1.2	13.08 ±1.5	6	7	16.02	0.62
3 actions	3	1331	3	16.71 ±2.1	14.67 ±3.2	15	11	46.76	29.45
4 actions	4	14641	4	15.02 ±1.9	10.27 ±3.6	16	159	108.22	1180.25

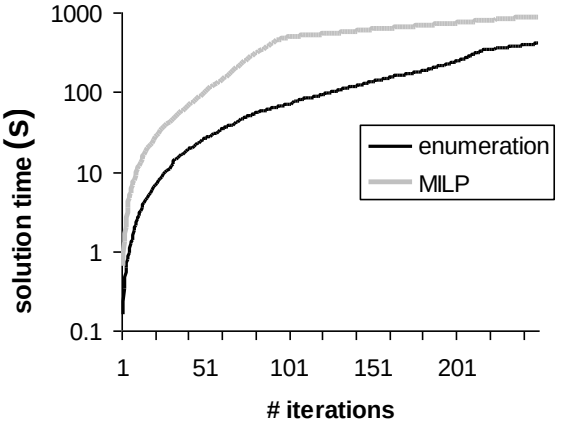
6.5.2 POMDP with discrete but large action space – A network flow problem

The general structure of a network consists of nodes and edges. The nodes shown as circles in Figure 6.8, facilitate the accumulation, production or consumption of materials or information while the edges facilitate flow of these quantities from one node to the other. The edges are shown as arrows originating from the source node and pointing to destination or recipient node. Many studies (Berry 2005; Rico-Ramirez et al, 2007) have been conducted for network design, i.e, determining the connectivity of the nodes. In other words, network design determines the existence of edges through which it is possible to transport material, information or both. The nodes of certain networks are amenable to contamination or corruption, e.g., computer networks can get infected with virus/bugs, food and water networks are prone to chemical or biological contamination and electrical networks are amenable to outages. Aside from origination, the contamination or infection spreads as the material or data flows from one node to the other. Therefore, it is imperative to track down the infected node and redirect the network flows. To this end, sensor network design studies have been performed to determine optimal locations to install measurement sensors. However, in the face of budget constraints all the nodes and edges cannot be inspected.

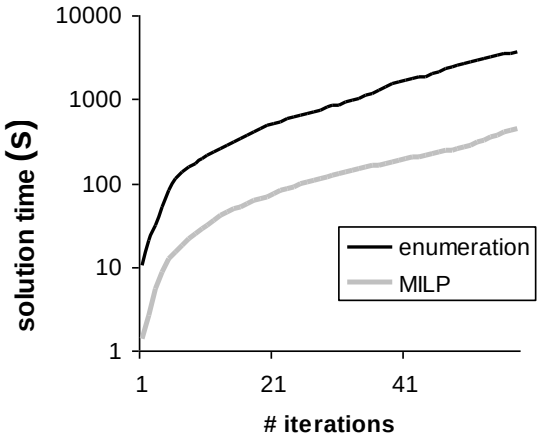
(a)


1000
100
10
1
0.1
0.01
0.001
solution time (s)
enumeration
MILP
1
21
41
61
81
101
iterations

(b)


1000
100
10
1
0.1
solution time (s)
enumeration
MILP
1
51
101
151
201
iterations

(c)


10000
1000
100
10
1
solution time (s)
enumeration
MILP
1
21
41
iterations

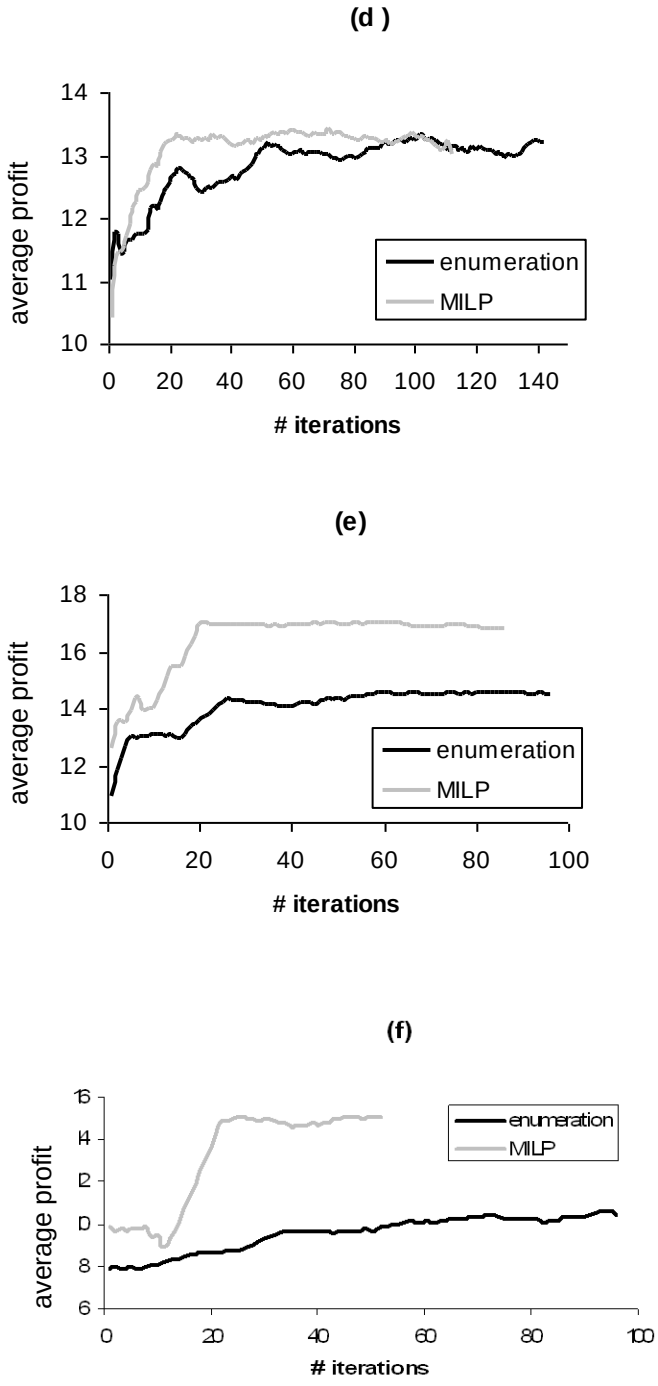


Figure 6.7. The comparison of convergence times and performance for the problem with continuous actions. Number of iterations v/s solution time for the problem with (a) two continuous actions, (b) three continuous actions, (c) four continuous actions. Average profit as a function of number of iterations for the problem with (d) two continuous actions, (e) three continuous actions, (f) four continuous actions.

Whereas network design and sensor allocation are one time decisions requiring substantial investments, network flow decisions are dynamic decisions which must be taken at each time so as to minimize the expected spread of the contaminant. The network flow decisions differ from network design decisions in that the latter determine which two nodes *can* have flow/connectivity between them by the presence of an edge, while the flow decisions determine whether or not to use an existing edge for flow. Especially in computer, electrical and water networks the alteration in network flows is speedy and comes at very low cost.

Figure 6.8 shows an instance of aforementioned network flow problem with five nodes (η_1, η_2, η_3, η_4, η_5). The network is fully connected in that all nodes are connected to each other except node η_1. η_1 serves as the source node which has no incoming streams. Measurement sensors are installed at nodes η_2 and η_3. This can be interpreted as saying that all outgoing streams emanating from nodes η_2 and η_3 are tested. The measurement is prone to type I errors (Lee and Unnikrishnan, 1998), such that the presence of a contaminant is detected with probability 0.9. If a node is connected to an upstream node, a reward of C_P units weighted by the population density at that node is received. However, if the node is infected, no reward is received. There are two ways in which a node can get infected: (i) origination of infection at that node and (ii) propagation of contaminant from an upstream node. Contamination is originated at a node with predetermined probabilities p_i for i=2,3,4,5. At each time, at most one node may be infected by origination of contaminant at that node. Alternatively, if an upstream node is infected, the recipient node would get infected at the next time period. It is assumed that the source node η_1 is never contaminated.

A node once contaminated, remains so unless a clean-up is performed. The clean-up has an associated cost C_r which is assumed to be independent of the node being cleaned. Finally, there is a cost $C_{flow}(i,j)$ associated with flow of material/information in the network. The cost depends on the source node i and recipient node j. The objective is to determine the optimal flow configuration and clean-up strategy at each time, so as to maximize average (infinite horizon discounted) profit. The knowledge of the nodes being contaminated is not complete.

We begin with a POMDP formulation of the network flow problem described above, and analyze the problem size. This is followed by the MILP formulation of the value backup operation.

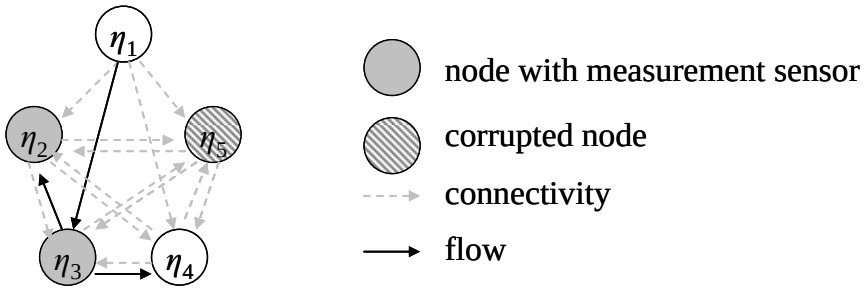


Figure 6.8: A five-node network flow problem

A network security example is presented in Figure 6.8, where the circles represent the nodes of the network and the arrows represent the flow of material, information etc. The shaded circles represent ones with measurement sensors and hashed circles indicate corrupted nodes. The light dashed lines show network connectivity and solid lines show a flow configuration.

Formulation as POMDP

While the state and action spaces are easily represented, the transition function is complicated for this problem. The POMDP model is shown below:

State

$$s(i,t) \in \{0,1\} \quad i = 2,3,..,N$$

$$|S| = 2^{(N-1)} \tag{6.38}$$

The size of state space is given by (6.38) where N is the number of nodes in the network

Action

$$a_{flow}(i, j, t) \in \{0,1\} \quad i = 1,2,.., N; \quad j = 2,3,.., N$$
$$a_{repair}(i, t) \in \{0.1\} \quad i = 2,3,..., N$$

The term $a_{flow}(i,i)$ for i=2,3,..,N is the node activation term. $a_{flow}(i,i)$=1 implies that node i is connected indicating that it has a positive inflow from some other node in the network. $a_{flow}(1,1)$ is always 1.

$$|A| = 2^{N(N-1)}+2^{(N-1)} \tag{6.39}$$

The expression for the size of the action space in (6.39) suggests a very large action space for even small values of N. However, many actions are not feasible. E.g.

(i) A node with no inflow cannot have any outflow or accumulation
(ii) A node with positive inflow must have either accumulation or outflow
(iii) A node can be either upstream or downstream of another node but not both.
(iv) Each node must have at most one supply node.

The above constraints reduce the size of the action space considerably.

State transition function

Given the state and action at the current time, the state at the next time may be determined by the following relations:

(i) Contamination by origination – A node i is contaminated by origination depending on the contamination probability p_i. This is shown in (6.40), where $p(t+1)$ is the realization of probability of origination of contamination and $\delta(.)$=1 if the condition within the parenthesis is met.

$$s(i, t+1) = \delta(p(t+1) \geq p_i) \quad for\ i = 2,3,..,5 \tag{6.40}$$

(ii) Contamination by propagation – If a source node *k* is contaminated, recipient node *i* will contaminated at the next time period (6.41).

$$s(i,t+1) = s(k,t)a_{flow}(k,i,t) \quad for\ i = 2,3,..,5;\ k = 2,3,..,5 \tag{6.41}$$

(iii) Carried over contamination – If node *i* is contaminated and not repaired at the current time, it will be contaminated at the next time period (6.42).

$$s(i,t+1) = \max\{s(i,t) - a_{repair}(i,t),0\} \quad for\ i = 2,3,..,5 \tag{6.42}$$

The state transition matrix for contamination by origination is a static matrix independent of current state and action. This constant matrix is designated as map_1. The transition matrices that account for points (ii) and (iii) above need to be determined for each state action pair. The matrix that combines the effects of factors (ii) and (iii) is designated by map_2. Therefore, the post decision belief state is given by (6.43).

$$b^a(s') = \sum_{s \in S} b(s) map_1(s,s') map_2^{\{a\}}(s,s') \tag{6.43}$$

Observation space and observation probability matrix

$$o(i,t) \in \{0,1\} \quad i \in \mathrm{M}$$

$$|O| = 2^{|M|} \tag{6.44}$$

The size of observation space is given by (6.44) where M is the set of nodes with measurement sensors. It is assumed that all measurement sensors can sense the presence of a contaminant 90% of the time. The observation probability matrix reflects this fact. E.g. if the measurement sensors are installed at nodes η2 and η3, the possible observations are:

$$\{o(2,t), o(3,t)\} = \{0,0\};\ \{1,0\};\ \{0,1\};\ \{1,1\}$$

When nodes η_2 and η_3 are both contaminated, i.e., $s(2,t)=s(3,t)=1$, the observation probabilities are given by:

$$p(\{o(2,t),o(3,t)\}=\{0,0\})=(1-0.9)^2$$
$$p(\{o(2,t),o(3,t)\}=\{1,0\})=0.9(1-0.9)$$
$$p(\{o(2,t),o(3,t)\}=\{0,1\})=(1-0.9)0.9$$
$$p(\{o(2,t),o(3,t)\}=\{1,1\})=0.9^2$$

The observation probabilities for the other states may be calculated in a similar manner.

Profit/reward Function

The profit function at time t comprises of the following terms

(i) Reward for active non-contaminated nodes – This is given by the first term in (6.45)

(ii) Cost of flow and repair actions – This is given by the second and third terms in (6.45).

$$r_t(s,a)=C_P\sum_i w_i a_{flow}(i,i,t)s(i,t)-\sum_j\sum_i C_{flow}(i,j)a_{flow}(i,j,t) - \sum_i C_{repair}(i)a_{repair}(i,t) \quad (6.45)$$

The value backups using enumeration are straightforward once the transition matrices are developed for each action. The MILP for value backups, however, is more involved. This is discussed below:

MILP based value backups

The MILP formulation must account for the following

(i) The constraints to exclude infeasible actions

(ii) The equations to develop state transition matrix map_2 as a function of actions.

Whereas the constraints to exclude infeasible actions (described earlier in this section) are relatively easily developed, the second set of constraints and equations to generate map_2 is a more difficult task. The MILP backup for a post-decision belief state $\tilde{b}^a$ is developed by using its predecessor pre-decision belief state, for each of the possible observations $b^{a,o} \forall o$. For a given *o*, let $b = b^{a,o}$ and let $\tilde{b}^{a'}$ denote the next post-decision belief state. The following sequence of events is followed:

(i) At the beginning of time *t*, the repair decision is implemented. The system transitions from state *s* to an intermediate state *s*1.

(ii) The node connectivity is determined and the appropriate reward generated.

(iii) Then the flow decisions are taken and the state of the system at the next time is s_2. This is because it is assumed that a contaminated node corrupts a downstream at the next time.

(iv) Finally, the contamination probabilities are realized and the cycle is repeated.

The MILP model for this process is shown in Figure 6.9. M3.1 through M3.3 determine the value function for $\tilde{b}^{a'}$, M3.4 and M3.5 model the input-output constraints described before and M3.6, M3.7, .. are the constraints that ensure that there is no cyclical flow between any two, three,..*N* nodes. The equations are only shown for two and three node combinations. This is similar to saying that a node cannot supply and receive material from another node at the same time.

A variable *z*(*l*,*i*) is introduced to denote the status of connectivity of node *i* in state *s*1(*l*,*i*) where *l*=1,2,..|*S*|. *z*(*l*,*i*) is determined by M3.8 and M3.9. The effects of repair and flow actions on state transitions are determined by M3.10 through M3.12. Having found a state *s*2(*l*,*i*) corresponding to each state *s*(*l*,*i*) in the state space, the key is to develop a map for the transition in order to determine the belief state $b^{a'}$. This is determined by obtaining the index of *s*2 within the state space with the help of variables $\tau.\tau(l,l1,i) = 0 \ \forall i$ when *s*2(*l*,*i*) is equal to index *l*1 of the state space. This implies that state designated by *l* transitions to *l*1. Correspondingly, map_2(*l*,*l*1) is set to 1. This is

model M3

$$\max \sum_{i,l} b(l)\{C_P w(i) z(l,i) - C_{repair}(i) a_{repair}(i)\} - \sum_{i,j} C_{flow}(i,j) a_{flow}(i,j) - 100 \sum_{l,l1,i} t(l,l1,i) + 0.9v$$

s.t.

$$v \geq \sum_{l} \tilde{b}^{a'}(l)\alpha(m,l) \qquad \forall m \qquad \text{M3.1}$$

$$v \leq \sum_{l} \tilde{b}^{a'}(l)\alpha(m,l) + M\,2(1 - y(m)) \qquad \forall m \qquad \text{M3.2}$$

$$\sum_{m} y(m) = 1 \qquad \text{M3.3}$$

$$\sum_{j} a_{flow}(j,i) - a_{flow}(i,i) \geq a_{flow}(i,i) \qquad i = 2,3,...N \qquad \text{M3.4}$$

$$\sum_{j} a_{flow}(j,i) - a_{flow}(i,i) \geq a_{flow}(i,k) \qquad i = 2,3,...N, \forall k \qquad \text{M3.5}$$

$$a_{flow}(i,j) + a_{flow}(j,i) \leq 1 \qquad \forall i \neq j \qquad \text{M3.6}$$

$$a_{flow}(i,j) + a_{flow}(j,k) + a_{flow}(k,i) \leq 2 \qquad \forall i \neq j \neq k \qquad \text{M3.7}$$

....

$$z(l,i) \leq a_{flow}(i,i) \qquad \forall l,i \qquad \text{M3.8}$$

$$z(l,i) \leq 1 - s1(l,i) \qquad \forall l,i \qquad \text{M3.9}$$

$$s1(l,i) \geq s(l,i) - a_{repair}(i) \qquad \forall l,i \qquad \text{M3.10}$$

$$s2(l,i) \geq s1(l,i) \qquad \forall l,i \qquad \text{M3.11}$$

$$1 - s2(l,k) \leq 1 - s1(l,i) + 1 - a_{flow}(i,k) \qquad \forall l, i \neq k \qquad \text{M3.12}$$

$$\sum_{i} \tau(l,l1,i) \geq s2(l,i) - s(l1,i) \qquad \forall l,l1 \qquad \text{M3.13}$$

$$\sum_{i} \tau(l,l1,i) \geq -1\{s2(l,i) - s(l1,i)\} \qquad \forall l,l1 \qquad \text{M3.14}$$

$$map_2(l,l1) \geq 1 - \sum_{i} \tau(l,l1,i) \qquad \forall l,l1 \qquad \text{M3.15}$$

$$\sum_{l1} map_2(l,l1) = 1 \qquad \forall l \qquad \text{M3.16}$$

$$\tilde{b}^{a'}(l2) = \sum_{l,l1} b(l) map_2(l,l1) map_1(l1,l2) \qquad \forall l \qquad \text{M3.17}$$

Figure 6.9: MILP model for the value backup for network flow POMDP

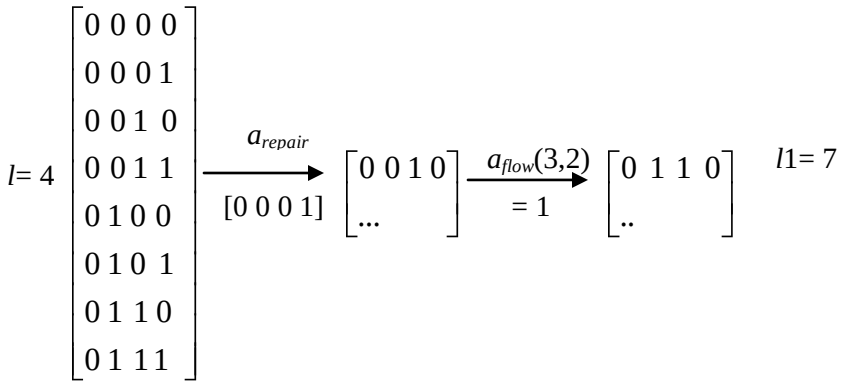


Figure 6.10. Illustration for mapping state l to $l1$

Table 6.3. Parameter values for the network flow problems

Parameter Description	**symbol**	**Value**	***i*=1**	***i*=2**	***i*=3**	***i*=4**	***i*=5**	***i*=6**
Origination Probability for six-nodes problem	p_i		0	0.04	0.025	0.015	0.01	0.01
For five-nodes problem	p_i		0	0.04	0.025	0.015	0.02	
For five-nodes problem	p_i		0	0.04	0.025	0.035		
Weighting factor	w_i		-	6	4	7	6	8
Repair cost per node	C_{repair}	30						
Reward for each active node	C_P	1						

$$C_{flow} = \begin{bmatrix} 0 & 0.1 & 0.2 & 0.5 & 1 & 0.5 \\ 10 & 0.1 & 0.1 & 0.5 & 0.8 & 0.3 \\ 10 & 0.1 & 0 & 0.1 & 0.1 & 0.1 \\ 10 & 0.5 & 0.1 & 0 & 0.5 & 1 \\ 10 & 0.8 & 0.5 & 0.1 & 0 & 0.8 \\ 10 & 0.7 & 1 & 0.2 & 0.4 & 0.1 \end{bmatrix}$$

Figure 6.11 Cost parameter for the network flow problem with four, five and six nodes

achieved by equations M3.13 through M3.16 and the concept is further illustrated by Figure 6.10. Once map_2 is determined, $b^{a'}$ is calculated using M3.17.

The network flow problem described above is solved for four, five and six nodes. In all three cases, nodes η_2 and η_3 have measurement sensors and the parameter values are shown in Table 6.3 and Figure 6.11. The solution for the three problem instances is shown in Table 6.4. The performances are comparable for the two methods. This is because the exact same problems are being solved in this case as opposed to using a smaller action space in the previous example. The solution times on the other hand follow a similar trend as the previous example in that it increases very rapidly with size for the enumeration based solution methods. Since $|V_n|$ is similar for the two solution approaches (in all three cases), the difference in solution times is almost entirely attributed to exponentially increasing action space due to combinatorial reasons while using enumeration. The behavior of the solution times is better understood by Figures 6.12 (a),(b) and (c). It is observed that both for MILP and enumeration based methods, the number of iteration v/s solution time has three distinct regions:

(i) For the first few iterations, the cumulative solution time is linear with a very small slope.

(ii) After this, the solution time grows exponentially with iteration count.

(iii) Finally, it becomes linear again with a large slope.

This is because the value function is initialized with a single gradient vector and the value function is set to the lowest possible reward. Due to this reason, a single update improves the value function for the entire sampled belief set. Consequently, only a single gradient vector comprises the value function for the first few iterations leading to same solution time per iteration. Beyond this point, the size of the value function, i.e., the number of gradient vectors during each iteration increase as the shape of the value function begins to take form. This causes the solution time to increase per iteration, as more gradient vectors are added. Finally, the value function converges and no new gradient vectors are added. At this point, the solution time again becomes constant for each iteration. The time per iteration is much higher as compared to that for the initial iterations.

Finally, the convergence of value function is depicted in Figure 6.13 (a) and (b) for the four and five node problems respectively. The average performance is plotted as a function of solution time. For the four node network flow problem, the value function converges faster for the enumeration based method. But the phenomenon is reversed for the five node problem. As seen from Table 6.4, the performance of the MILP based method only gets better with increase problem size.

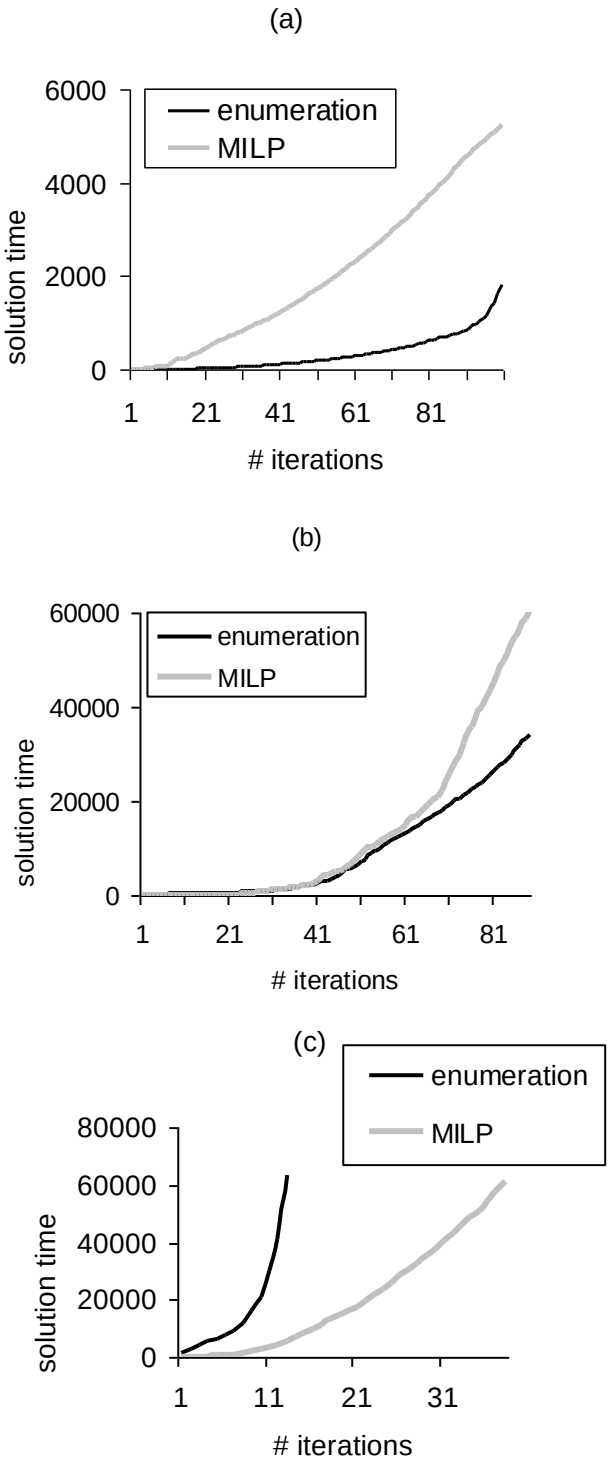


Figure 6.12: Comparison between solution times of the MILP based backups and the enumeration based backups for the network flow problem with (a) four nodes, (b) five nodes and (c) six nodes.

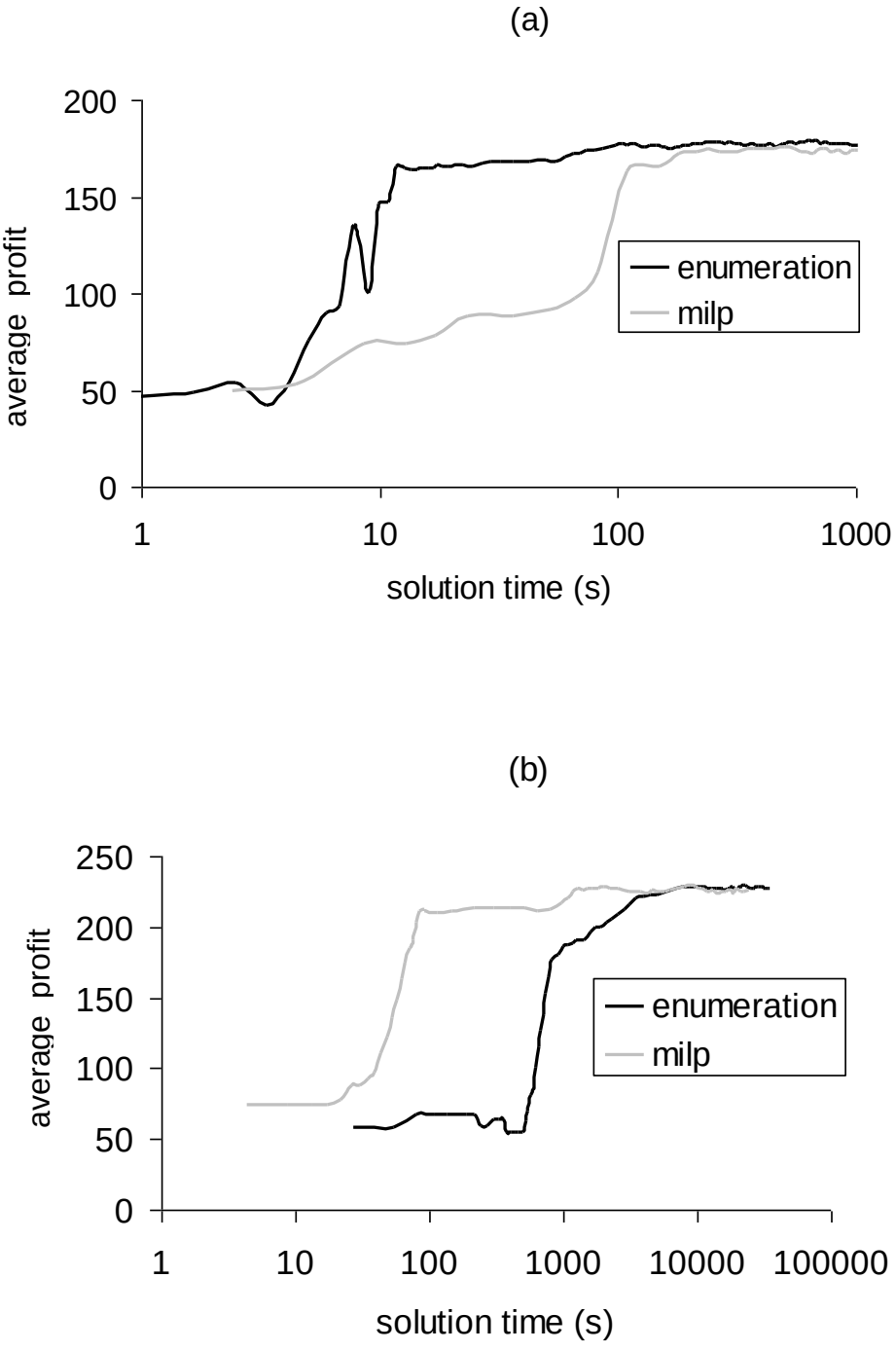


Figure 6.13: Comparison of convergence and performance of the MILP and enumeration approaches for the network flow problem with (a) four nodes, (b) five nodes.

Table 6.4: Problem size and results for the network flow problems.

Problem	$\|S\|$	$\|A\|$	$\|O\|$	performance MILP	performance enum	$\|V_n\|$ MILP	$\|V_n\|$ enum	Convergence time MILP	Convergence time enum
Network flow									
4 nodes	8	2336	4	179.54± 13.56	181.22 ± 12.51	22	39	174	97
5 nodes	16	15296	4	225.91± 21.45	223.96 ± 25.76	42	43	1168	3132
6 nodes	32	63744	4	292.51± 10.72	292.67± 9.94	73	67	5641	63279

6.6 Conclusions

In this chapter, MILP models are developed to obtain value backups for POMDPs with very large or continuous action spaces. The traditional way of doing this is by using either a homogeneous or heterogeneous grid or sampling methods. By using MILP based methods, we can ensure exact calculation of the max operator for each value backup, thereby preserving the solution quality. The algorithm in its current form, can solve POMDPs with discrete state spaces and when the observation probabilities are independent of continuous actions. Certain restrictions are also applied on the state transition and reward functions to maintain the linearity of the value backup operation.

The MILP is amenable to large computation times when the size of the observation space is large. For this reason, an alternative MILP model is developed for solution of POMDP around post decision belief state. This removes the dependence of MILP solution time on the size of the observation space. Two examples are presented to illustrate the solution method, one each for value updates around pre and post-decision belief states. It is observed that the MILP based method produces higher quality result,

when the entire action space is not considered for enumeration based method. The solution gap increases with problem size.

Additionally, the solution time scales linearly with problem size for the MILP based method, while it grows exponentially in case of enumeration based methods. This is attributed to the exponential increase in the size of action space due to combinatorial reasons. Consequently, the MILP quickly surpasses the enumeration based methods in terms of lower solution times and better solution quality as the problem size is increased. This makes the MILP based solution very well suited for POMDPs with high dimensional actions and continuous actions.

CHAPTER 7

CONTRIBUTIONS AND FUTURE WORK

7.1 Conclusions and Contributions

Optimal resource management is an integral part of any manufacturing environment. This body of work is a step forward in devising better resource management policies in conjunction with production, inspection or distribution strategies. This is performed for three different types of resource degradation models spanning many process industries. From an academic standpoint, this thesis seeks to make contributions in the following areas:

Introduction of a new class of problems

In Chapters 3 and 4, a new class of problems, the inventory control of *perishable resources* is considered. The study is aimed at resources that get consumed, break or exit the system, with time scales comparable to that of production. Aside from small and specialized manufacturing equipment, the technique can be extended to human resource management in the form of hiring and training decisions.

The inventory of perishable resources is directly affected by two important factors: (i) the procurement policy which adds to the inventory (ii) the production process, whereby the resources age and break, resulting in depletion of inventory. It is obvious then, that the production decisions directly affect the resource management decisions. The availability of resources, on the other hand, place constraints of production quantities. These inter-dependent decisions are considered as a part of a combined optimization problem presented in Chapter 3. It is shown by numerical examples that the performance is significantly better when the resource management decisions are considered together with production planning as opposed to independent of it. The performance gap depends on the level to which the resource breaking rate is affected by

usage, average resource life and variability in the total number of resources used per time period.

When the system is plagued by high level of uncertainty, the problem is reformulated as an MDP and solved using approximate dynamic programming (ADP) in Chapter 4. For the deterministic problem and for the parameter set considered, the solution obtained by the ADP algorithm is 94.5% of the optimal solution. For the stochastic demand and resource life models, the ADP algorithm proves to be a more flexible and widely applicable choice. This is because the ADP algorithm is capable of taking a more comprehensive view of the uncertainty, thereby taking improved and more informed decisions as compared to the rolling horizon methodology.

Practical extension to an existing problem

Randomly deteriorating machines with limited observability, have been traditionally solved as partially observable Markov decision processes. In Chapter 5, this work is extended by considering multiple machines and multiple types of jobs being operated on the machine. The study is motivated, in part, by the recent advancements in efficient solution of medium to large size POMDPs.

Costly inspections dissuade the decision-maker to test all the processed jobs generating a possibility of propagation and accumulation of defective items. Therefore, information on defective items needs to be maintained at all times. This results in a significantly larger state space and consequently, the overall POMDP. An approximate solution method called PERSEUS (Spaan and Vlassis, 2005) is used for the solution of POMDPs. Structure of the optimal policy and that of the value function is studied. While the treatment of the above problem as a POMDP is shown to offer promise in terms of better performance over heuristic rules, the characterization of either the optimal policy or value function or both is required, so that solutions to problems of practical scale can be obtained. This point is discussed further in section 7.2.

Algorithmic Development

While point based algorithms for solution of POMDPs prove efficient with discrete or small action spaces, literature on large action spaces is limited to action-

sampling based algorithms. Whereas, this might be a successful strategy for some problems, it is difficult to devise sampling schemes for many applications. In Chapter 6, we introduce the notion of obtaining Bellman updates or value backups using a mixed integer mathematical programming (MILP) model. The MILP based backups are made possible by the piecewise linear and convex value function and certain requirements on reward and probability functions. The latter essentially ensure a linear model.

Recognizing that a large size observation space significantly increases the solution time of MILP, we present an alternative formulation of POMDP around post-decision belief state. In the latter case, dependence of MILP on the size of observation space is removed.

The algorithmic strategy is tested on two example problems, one with continuous actions and the other with discrete but very large action space. It is seen that the MILP based approach quickly surpasses the enumeration based methods in terms of lower solution times and/or better solution quality as the problem size is increased. This makes the MILP based solution very well suited for POMDPs with high dimensional actions and continuous actions.

7.2 Future Work

There are several avenues along which the current work can be improved or extended. Some of these are noted below:

Macroeconomic objectives

The combined decision-making in all the problems is performed with the objective of maximizing a measure of profit. In manufacturing environments, typical objectives aside from maximizing profit are maximizing cycle time, maximizing throughput or minimizing tardiness. These objectives would require other considerations, thereby making the problem richer and more practical.

Characterization of optimal policy

For inventory control problems as well as maintenance problems for a single randomly deteriorating system, the existence of parametric form of optimal policy has been proven (given that certain conditions are met). It has been shown that for the inventory control problem and fully observable maintenance problem, the optimal policy has a control limit structure. For the partially observable machine maintenance problem, the optimal policy is marginally monotone in the belief space. These findings simplify the search for optimal policy parameters for large size problems, without having to solve the actual optimization problem using rigorous methods.

For the extensions presented in this work, it may be possible to find simplified instances or subclasses of problems for which characterization of optimal policy using a parametric form can be performed. Additionally, the structure of the value function for the (single) machine maintenance problem is shown to be marginally monotone (when certain conditions are met). A good strategy will be to add features to the single machine problem so that the form of the value function is preserved. Problems developed using this bottom-up approach would enable use of faster algorithms that take advantage of the known structure of the value function.

Improvements in MILP based updates for POMDPs

Since an MILP is solved for each value backup, it is required that the model be computationally sound. In Chapter 6, the MILP formulation involves the big-M type constraints which potentially slow down the computation. A stronger formulation may result in significant improvements in MILP solution time.

Continuous state spaces

Presence of continuous actions in a POMDP is often coupled with continuous states. The algorithm developed in Chapter 6 can only work with discrete states. In point based algorithms, continuous states are handled using Gaussian mixtures (Spaan, 2006) in the spirit similar to particle filters. Further research is required to determine if the MILP based backups may be extended to POMDPs with continuous states

APPENDIX A

THEOREM 1

The theorem below is stated without proof. The proof can be found in (Monahan, 1982) or (Ivy, 2005)

If for $0 \le b(j) \le 1$ and j=1,2..N-1

(i) $Q(b,a)$ is a non-decreasing or non-increasing function of $b(j) \forall a$

(ii) $Q(b,a) - Q(b,a')$ is a non-decreasing or non-increasing function of $b(j)$ for $a < a'$

Where '<' denotes a partial ordering of actions,

Then the optimal policy is marginally monotone in the belief state $b(s), s = 1,2,..., N$

APPENDIX B:

DEMONSTRATION FOR MONOTONE VALUE FUNCTION

To prove:

Given, $0 \le b(j) \le 1$ for j=1,2..N-1

$Q(b,a)$ is a non-decreasing or non-increasing function of $b(j)$:

$$\frac{\partial Q(b,a)}{\partial b(j)} \ge (\le) 0 \ \forall a,\ j = 1,2,..,N-1$$

$$V(b) = \max_{a \in A} Q(b,a) \quad \Rightarrow \frac{\partial V(b)}{\partial b(j)} \ge (\le) 0 \ \ for\ j = 1,2,...N-1$$

Proof:

Recall that the value function $V(b)$ is the optimal infinite horizon discounted reward. We will prove this using mathematical induction. Let k denote the stage of the MDP

1. For k=1, show that $\frac{\partial Q^k(b,a)}{\partial b(j)} \ge (\le) 0 \ \forall a$
2. For an arbitrary m>1 and k=m, assume $\frac{\partial Q^k(b,a)}{\partial b(j)} \ge (\le) 0 \ \forall a$
3. For k=m+1, show that $\frac{\partial Q^k(b,a)}{\partial b(j)} \ge (\le) 0 \ \forall a$

Also recall some system properties:

$(i)\ r(s,a) - r(s1,a) \geq 0 \quad for\ \ s < s1$

$(ii)\ p(s'|\, s,a) - p(s'|\, N,a) \geq 0 \quad for\, s' \neq N$

$(iii)\ p(o_1 \mid s,a) - p(o_1 \mid s1,a) \geq 0 \quad for\, s < s1$

$(iv)\ \ p(o_1 \mid s=1,a) \geq p(o_2 \mid s=1,a) \ \ \forall a$

where o_1 is the observation that no defect has occurred and o_2 is the observation that defect has occurred

For *k*=1

$$Q^1(b,a) = \sum_{s=1}^{N} r(s,a)b(s) \quad \forall a$$

since $\sum_{s=1}^{N} b(s) = 1, \quad b(N) = 1 - \sum_{s=1}^{N-1} b(s) = 1$

$$\frac{\partial Q^1(b,a)}{\partial b(j)} = r(j,a) - r(N,a) \geq 0 \ \ \forall a,\ j = 1,2,...,N-1 \qquad \text{by (i)}$$

$$\Rightarrow \frac{\partial V^1(b)}{\partial b(j)} \geq 0 \qquad \text{for } j=1,2,..N\text{-}1$$

For *k*=*m*, assume

(v) $\quad \dfrac{\partial Q^m(b,a)}{\partial b(j)} \geq 0 \ \ \forall a,\ j = 1,2,...N-1$

Then,

For *k*=*m*+1

$$Q^{m+1}(b,a) = \sum_{s=1}^{N} r(s,a)b(s) + \gamma \sum_{o} \sum_{s'=1}^{N} \sum_{s=1}^{N} \{ p(o \mid s',a)\, p(s'|\, s,a) b(s) V^m(b^{a,o}(s')) \}$$

where,

$$b^{a,o}(s') = \frac{p(o \mid s',a) \sum_{s=1}^{N} p(s'|\, s,a) b(s)}{\delta^o}$$

$$\delta^o = \sum_{s'=1}^{N} p(o \mid s',a) \sum_{s=1}^{N} p(s'|\, s,a) b(s)$$

Using the chain rule for differentiation

$$\frac{\partial Q^{m+1}(b,a)}{\partial b(j)} = r(j,a) - r(N,a) + \gamma \sum_{o} \sum_{s'=1}^{N} \sum_{s=1}^{N-1} \{p(o \mid s',a)\, p(s' \mid s,a) b(s) \frac{\partial V^{m}(b^{a,o})}{\partial b(j)}\} +$$
$$\gamma \sum_{o} \sum_{s'=1}^{N} \{p(o \mid s',a)\, p(s' \mid j,a) V^{m}(b^{a,o}(s'))\} +$$
$$\gamma \sum_{o} \sum_{s'=1}^{N} \{p(o \mid s',a)\, p(s' \mid N,a) b(N) \frac{\partial V^{m}(b^{a,o})}{\partial b(j)}\}$$

Rearranging terms,

$$\frac{\partial Q^{m+1}(b,a)}{\partial b(j)} = r(j,a) - r(N,a) +$$
$$\gamma \sum_{o} \sum_{s'=1}^{N} [\, p(o \mid s',a) \sum_{s=1}^{N-1} \{\{p(s' \mid s,a) - p(s' \mid N,a)\} b(s) + p(s' \mid N,a)\} \frac{\partial V^{m}(b^{a,o})}{\partial b(j)}\}] +$$
$$\gamma \sum_{o} \sum_{s'=1}^{N} \{p(o \mid s',a) \{p(s' \mid j,a) - p(s' \mid N,a)\} V^{m}(b^{a,o}(s'))\}$$

(A) (B) (C)

(vi)

Also, from chain rule,

$$\frac{\partial V^{m}(b^{a,o})}{\partial b(j)} = \frac{\partial V^{m}(b^{a,o})}{\partial b^{a,o}(s')} \frac{\partial (b^{a,o}(s'))}{\partial b(j)}$$

$$\frac{\partial b^{a,o}}{\partial b(j)} = \frac{p(o \mid s',a)\{p(s' \mid j,a) - p(s' \mid N,a)\}}{\delta^{o}}$$

By (*i*), term $A \geq 0$

Since the probabilities are non-negative and $p(s' \mid s,a) - p(s' \mid N,a) \geq 0$;

$$\Rightarrow \frac{\partial V^m(b^{a,o})}{\partial b^{a,o}(s')} \geq 0 \quad for \;\; s' \neq N \quad \text{from (v)}$$

Also, given $b(N) = 1 - b(1) - b(2)...$

$$\Rightarrow \frac{\partial V^m(b^{a,o})}{\partial b^{a,o}(s')} \leq 0 \quad for \;\; s' = N \quad \text{from (v)}$$

Note that term B (in (vi)) ≥ 0

For term C (in (vi)),

Substituting $p(s' \mid N, a) = 1 - \sum_{s=1}^{N-1} p(s' \mid s, a)$ and rearranging,

Term C= $\sum_{o} \sum_{s'=1}^{N-1} \{p(o \mid s', a) - p(o \mid N, a)\}\{p(s' \mid j, a) - p(s \mid N, a)\} V^m(b^{a,o}(s'))$

Now there are 3 possible observations:
o_1 – no defect
o_2 – defect
o_3 – no observation
o_3 is independent of state, therefore $p(o_3 \mid s', a) - p(o_3 \mid N, a) = 0$

Also, $p(o_2 \mid s', a) = 1 - p(o_1 \mid s', a) \quad \forall a, s$

$\Rightarrow$ Term C (in (vi))

$$= \sum \{p(o_1 \mid s', a) - p(o_1 \mid s', a)\}\{p(s' \mid j, a) - p(s' \mid N, a)\}\{V^m(b^{a,o1}(s')) - V^m(b^{a,o2}(s'))\}$$

By definition , $p(o_1 \mid s', a) - p(o_2 \mid N, a) \geq 0 \quad \forall a, s'$

and , $p(s' \mid j, a) - p(s' \mid N, a) \geq 0 \quad \forall a, s' \neq N$

Using the definition of $b^{a,o}$,

if $p(o_1 \mid s' = 1, a) \geq p(o_2 \mid s' = 1, a) \quad \& \; \frac{\partial V^m(b^{a,o})}{\partial b^{a,o}(s')} \geq 0$, it can be shown that

$V^m(b^{a,o1}(s')) \geq V^m(b^{a,o2}(s')) \quad for \;\; s' \neq N$

$\therefore$ Term C ≥ 0 and $\frac{\partial Q^{m+1}(b,a)}{\partial b(j)} \geq 0 \;\; \forall a, \; j = 1,2,..N-1$

Hence proved

APPENDIX C

DEMONSTRATION FOR MONOTONE POLICY

To prove: Given, $0 \le b(j) \le 1$ for j=1,2..N-1

$Q(b,a) - Q(b,a')$ for $a < a'$ is a non-decreasing or non-increasing function of $b(j)$:

$$\frac{\partial\{Q(b,a) - Q(b,a')\}}{\partial b(j)} \ge (\le) 0 \quad \text{for } a < a'$$

For *k*=1,

$$\frac{\partial\{Q^k(b,a) - Q^k(b,a')\}}{\partial b(j)} = r(j,a) - r(j,a') - r(N,a) + r(N,a') = 0 \quad \text{for } a < a'$$

There are three actions:

a_1 – do nothing

a_2 – job inspection

a_3 – machine maintenance

so we have 3 combinations for (a,a'), *i.e.*, $(a_1,a_2),(a_1,a_3),(a_2,a_3)$

Since state transition probabilities are independent of system state in case of a_3,

$$\frac{\partial Q^k(b,a_3)}{\partial b(j)} = 0 \quad \forall k .$$

From the proof in Appendix A,

$$\frac{\partial Q^k(b,a)}{\partial b(j)} - \frac{\partial Q^k(b,a_3)}{\partial b(j)} \ge 0 \quad for\, a = a_1, a_2,\ j = 1,2,..,\ N-1, \forall k$$

For *k*=*m*>1, assume

$$\frac{\partial E_o\{V^{m-1}(b^{a1,o})\}}{\partial b(j)} - \frac{\partial E_o\{V^{m-1}(b^{a2,o})\}}{\partial b(j)} \geq 0 \quad \forall o,\ j = 1,2,..,N-1 \qquad \text{(vii)}$$

where,

$$E_o\{.\} = \sum_o \{p(o)(.)\}$$

Note that the state transition probability matrix is the same. By (vii)

$$\Rightarrow \sum_{s'=1}^{N}\sum_{s=1}^{N-1}[\{p(s'|s,a) - p(s'|N,a)\}b(s) + p(s'|N,a)\{\frac{\partial V^{m-1}(b^{a1,o3})}{\partial b(j)} - \text{(D)}$$

$$p(o_1|s',a_2)\frac{\partial V^{m-1}(b^{a2,o1})}{\partial b(j)} - p(o_2|s',a_2)\frac{\partial V^{m-1}(b^{a2,o2})}{\partial b(j)}\}] +$$

$$\sum_{s'=1}^{N}\sum_{s=1}^{N-1}\{p(s'|s,a) - p(s'|N,a)\}\{V(b) - p(o_1|s',a_2)V^{m-1}(b^{a2,o1}) - p(o_2|s',a_2)V^{m-1}(b^{a2,o2}) \text{ (E)}$$

$$\geq 0 \quad \text{for } j = 1,2,..,N-1 \qquad \text{(viii)}$$

This implies that $\alpha_1 V_1^{m-1} + (1-\alpha_1)V_2^{m-1} \leq V_3^{m-1}$

where $\alpha_1 = p(o_1|s',a)$ and $V_i^{m-1} = V^{m-1}(b^{a,oi})$ for $i = 1,2,3 \Rightarrow$ positive term E (in (viii))

Taking derivative with respect to *b*(*j*),

$$\alpha_1 \frac{\partial V_1^{m-1}}{\partial b(j)} + (1-\alpha_1)\frac{\partial V_2^{m-1}}{\partial b(j)} \leq \frac{\partial V_3^{m-1}}{\partial b(j)} \quad \Rightarrow \text{positive term D (in (viii)).}$$

Given convexity and monotonicity of value function and since

$$r(s,a_1) > r(s,a_2)\ \forall s, \quad \alpha_1 V_1^m + (1-\alpha_1)V_2^m \leq V_3^m$$

$$\Rightarrow \frac{\partial E_o\{V^m(b^{a1,o})\}}{\partial b(j)} - \frac{\partial E_o\{V^m(b^{a2,o})\}}{\partial b(j)} \geq 0 \quad \forall o,\ j = 1,2,..,N-1$$

Hence proved

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