

C.4 Tree Diagrams and Bayes' Theorem

- Find probabilities using tree diagrams.
- Find probabilities using Bayes' Theorem.

Tree Diagrams

A type of diagram that can be used as an aid in computing probabilities is a **tree diagram**. For example, consider an experiment of tossing a six-sided die. Each time the experiment is repeated, the probability of obtaining a 1 (event A) is $P(A) = 1/6$. If you are only concerned with whether the number is 1 or not 1 (event A or event A'), and the experiment is repeated three times, then eight different sequences of events are possible. The tree diagram in Figure C.11 shows the probabilities of these eight sequences of events.

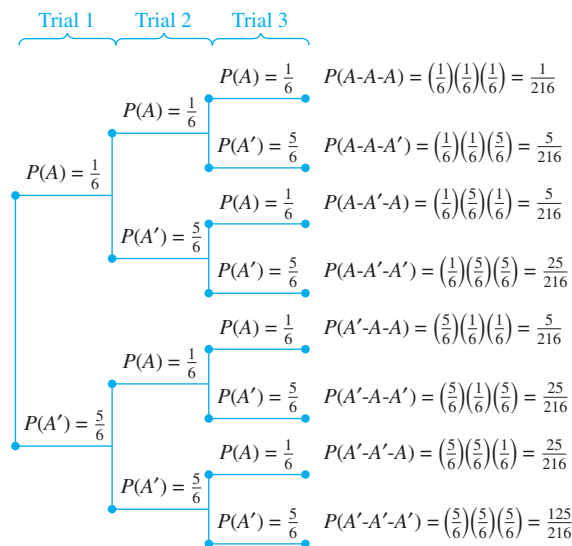


FIGURE C.11

Note that the probability of each sequence of events is given by the product of the probabilities of the events in the sequence.

If a tree diagram represents all possible sequences of events, as it does in Figure C.11, then the sum of the probabilities of the various sequences should be 1. For instance, the sum of the probabilities of the eight different sequences in Figure C.11 is

$$\frac{1}{216} + \frac{5}{216} + \frac{5}{216} + \frac{25}{216} + \frac{5}{216} + \frac{25}{216} + \frac{25}{216} + \frac{125}{216} = \frac{216}{216} = 1.$$

Tree diagrams often involve conditional probabilities, as demonstrated in the next two examples.

Example 1 A Tree Diagram Involving Conditional Probabilities 

A manufacturer orders a biomedical part from three different suppliers. From past records, the manufacturer finds that the three suppliers have the following percentages of defective parts.

	<u>Supplier 1</u>	<u>Supplier 2</u>	<u>Supplier 3</u>
Defective Parts	3%	5%	4%
Nondefective Parts	97%	95%	96%

The current inventory of parts contains 5000 units from Supplier 1, 3500 units from Supplier 2, and 2000 units from Supplier 3. If a part is chosen from the inventory at random, what is the probability that it is defective?

SOLUTION Because the current inventory contains 10,500 parts, the probability that a part chosen at random is from Supplier 1 is $5000/10500 = 10/21$, from Supplier 2 is $3500/10500 = 7/21$, and from Supplier 3 is $2000/10500 = 4/21$. In Figure C.12, let D represent the event that the chosen part is defective, and $S_1, S_2,$ and S_3 represent the events that the chosen part came from Suppliers 1, 2, and 3, respectively.

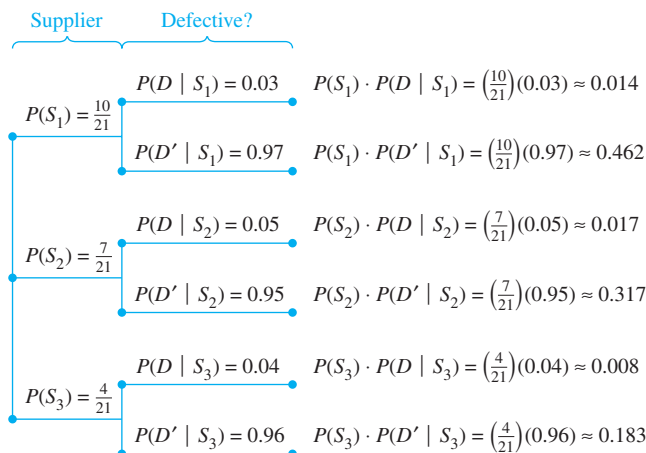


FIGURE C.12

From the tree diagram, you can see that the probability of choosing a defective part is

$$P(D) \approx 0.014 + 0.017 + 0.008 = 0.039.$$

 **CHECKPOINT 1**

In Example 1, suppose the current inventory of parts contains 4000 units from Supplier 1, 5000 units from Supplier 2, and 1500 units from Supplier 3. If a part is chosen from the inventory at random, what is the probability that it is defective? ■

Example 2 A Tree Diagram Involving Conditional Probabilities

A hospital has 300 nurses. During the past year, 48 of the nurses earned a pay raise. At the beginning of the year, the hospital offered a special training seminar, which was attended by 138 of the nurses. Twenty-seven of the nurses who earned a raise had attended the training seminar.

- What is the probability that a nurse who attended the seminar earned a pay raise?
- If a nurse is selected at random, what is the probability that the person attended the seminar *and* earned a pay raise?

SOLUTION Create a tree diagram showing the various probabilities, where event A represents attending the training seminar and event B represents earning a pay raise (See Figure C.13).

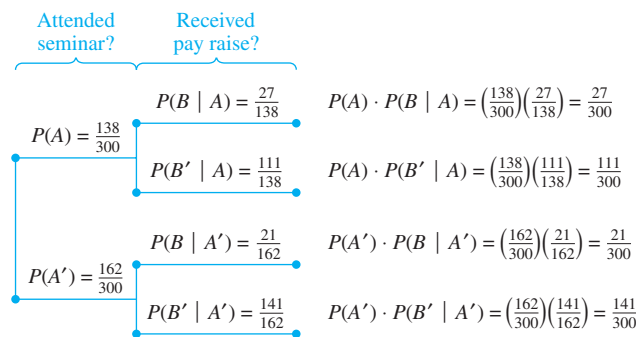


FIGURE C.13

- From the tree diagram, you can see that the probability that a nurse earned a pay raise, given that the person had attended the training seminar, is

$$P(B | A) = \frac{27}{138} \approx 0.196.$$

- If one of the nurses is selected at random, the probability that the person attended the training seminar *and* earned a pay raise is given by

$$P(A \cap B) = P(A) \cdot P(B | A) = \left(\frac{138}{300}\right)\left(\frac{27}{138}\right) = \frac{27}{300} = 0.09.$$

CHECKPOINT 2

A pharmaceutical company has 400 laboratory technicians. Recently, 50 of the technicians earned a bonus. At the beginning of the year, the company offered laboratory training which was attended by 210 of the technicians. Forty-two of the technicians who earned a bonus had attended the laboratory training. If one of the technicians is chosen at random, what is the probability that the person attended the laboratory training *and* earned a bonus? ■

Bayes' Theorem

Bayes' Theorem, named after the English mathematician Thomas Bayes (1702–1761), is an important formula that provides an alternative way of computing conditional probabilities.

Before the formula is given, take another look at a simple tree diagram involving two events A and B , as shown in Figure C.14.

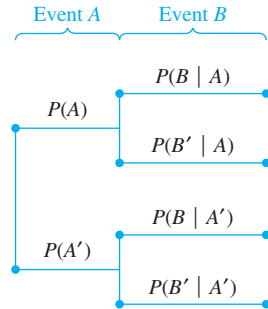


FIGURE C.14

From the diagram, the probability that event B will occur, given that event A has occurred, is known—this is a “forward-looking” probability in the sense that event A occurred before event B . Suppose instead that you were asked to find the “backward-looking” probability that event A has occurred, given that event B has occurred. In other words, you are asked to find

$$P(A | B).$$

Bayes' Theorem gives a way to find this conditional probability by using the formula

$$P(A | B) = \frac{P(A) \cdot P(B | A)}{P(A) \cdot P(B | A) + P(A') \cdot P(B | A')}.$$

Bayes' Theorem

The conditional probability that an event A has occurred, given that event B has occurred, is

$$P(A | B) = \frac{P(A) \cdot P(B | A)}{P(A) \cdot P(B | A) + P(A') \cdot P(B | A')}.$$

Example 3 Applying Bayes' Theorem

In Example 2, what is the probability that a nurse who earned a pay raise had attended the training seminar?

SOLUTION Figure C.15 shows the tree diagram for this problem, where event A represents attending the training seminar and event B represents earning a pay raise.

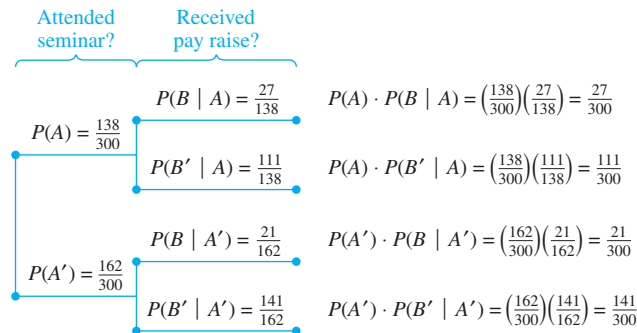


FIGURE C.15

Using Bayes' Theorem, the probability that a nurse attended the training seminar, given that the person earned a pay raise, is

$$\begin{aligned}
 P(A | B) &= \frac{P(A) \cdot P(B | A)}{P(A) \cdot P(B | A) + P(A') \cdot P(B | A')} \\
 &= \frac{(138/300)(27/138)}{(138/300)(27/138) + (162/300)(21/162)} \\
 &= \frac{27/300}{48/300} \\
 &= \frac{27}{48} \approx 0.563.
 \end{aligned}$$

This answer makes sense because you know from Example 2 that 48 people earned a pay raise, and of these, 27 had attended the training seminar.

CHECKPOINT 3

In checkpoint 2, what is the probability that a technician who earned a bonus had attended the laboratory training? ■

Bayes' Theorem can be generalized to include any number of mutually exclusive events whose union is the entire sample space. For instance, suppose in Figure C.16 that the events $A_1, A_2, A_3, \dots, A_n$ are mutually exclusive and that $A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n = S$. Then the conditional probability that the event A_1 has occurred, given that event B has occurred, is

$$P(A_1 | B) = \frac{P(A_1) \cdot P(B | A_1)}{P(A_1) \cdot P(B | A_1) + P(A_2) \cdot P(B | A_2) + \dots + P(A_n) \cdot P(B | A_n)}.$$

The formulas for $P(A_2 | B), P(A_3 | B), \dots$, and $P(A_n | B)$ are similar.

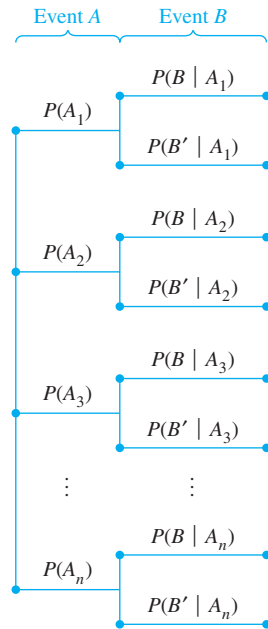


FIGURE C.16

Bayes' Theorem (General Case)

If $A_1, A_2, A_3, \dots, A_n$ are mutually exclusive and

$$A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n = S,$$

then the conditional probability that the event A_i has occurred, given that event B has occurred, is

$$P(A_i | B) = \frac{P(A_i) \cdot P(B | A_i)}{P(A_1) \cdot P(B | A_1) + P(A_2) \cdot P(B | A_2) + \dots + P(A_n) \cdot P(B | A_n)}.$$

Example 4 Applying the General Case of Bayes' Theorem ®

In Example 1, a manufacturer ordered biomedical parts from three different suppliers with the following percentages of defective parts.

	<u>Supplier 1</u>	<u>Supplier 2</u>	<u>Supplier 3</u>
Defective Parts	3%	5%	4%
Nondefective Parts	97%	95%	96%

If a part that is chosen at random from the inventory is found to be defective, what is the probability that it came from Supplier 1?

SOLUTION In the tree diagram shown in Figure C.17, let D represent the event that the chosen part is defective, and $S_1, S_2,$ and S_3 represent the event that the chosen part came from Suppliers 1, 2, and 3, respectively. Using the general form of Bayes' Theorem, the probability that event S_1 occurred, given that D occurred, is

$$\begin{aligned}
 P(S_1 | D) &= \frac{P(S_1) \cdot P(D | S_1)}{P(S_1) \cdot P(D | S_1) + P(S_2) \cdot P(D | S_2) + P(S_3) \cdot P(D | S_3)} \\
 &= \frac{\left(\frac{10}{21}\right)(0.03)}{\left(\frac{10}{21}\right)(0.03) + \left(\frac{7}{21}\right)(0.05) + \left(\frac{4}{21}\right)(0.04)} \\
 &\approx \frac{0.014}{0.039} \\
 &\approx 0.36.
 \end{aligned}$$

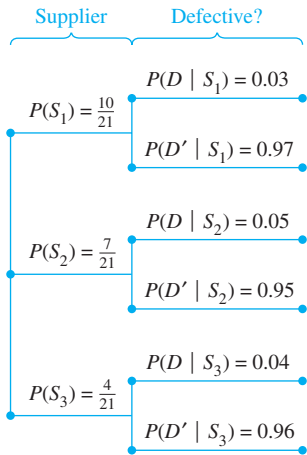


FIGURE C.17

✓ CHECKPOINT 4

In Example 4, suppose a part that is chosen at random from the inventory is found to be defective. What is the probability that it came from Supplier 2? ■

CONCEPT CHECK

1. According to _____, the probability of event A , given that event B has occurred, is

$$P(A | B) = \frac{P(A) \cdot P(B | A)}{P(A) \cdot P(B | A) + P(A') \cdot P(B | A')}$$

2. Describe how to find the probability of each sequence of events displayed within a tree diagram.
3. Can Bayes' Theorem be used to include multiple mutually exclusive events? Explain.
4. Describe the benefit of using tree diagrams to compute probabilities.

Skills Review C.4

The following warm-up exercises involve skills that were covered in earlier sections. You will use these skills in the exercise set for this section. For additional help, review Appendices C.1 and C.3.

In Exercises 1 and 2, a card is drawn from a standard deck of 52 playing cards.

1. What is the probability that the card is a face card or a 9?
2. What is the probability that the card is a jack or a king?

In Exercises 3 and 4, find $P(A | B)$.

3. $P(B) = 0.46, P(A \cap B) = 0.11$
4. $P(A) = 0.32, P(B) = 0.40, P(A \cup B) = 0.68$

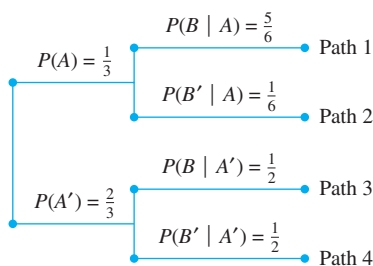
Exercises C.4

See www.CalcChat.com for worked-out solutions to odd-numbered exercises.

In Exercises 1–4, draw tree diagrams to represent the sequence of experiments.

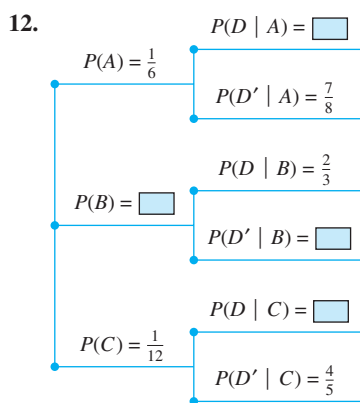
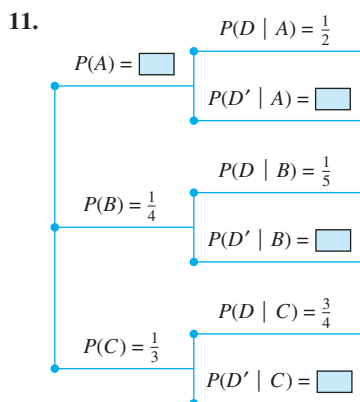
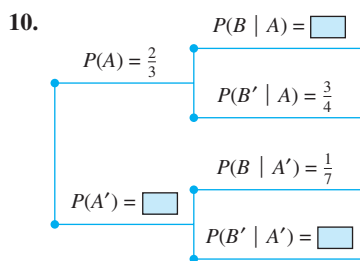
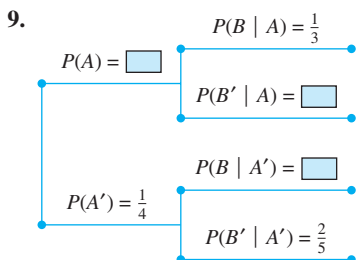
1. Two teams A and B play three games.
2. Three teams A, B, and C play two games.
3. An experiment with three equally likely outcomes is performed twice.
4. Three true or false questions are answered on a test.

In Exercises 5–8, use the following tree diagram to find the indicated probability.



5. Taking Path 1
6. Taking Path 2
7. Taking Path 3
8. Taking Path 4

In Exercises 9–12, complete the tree diagram.



Random Selection In Exercises 13–16, there are 9 gold keys and 12 silver keys. Four of the gold keys and six of the silver keys are used in Building 1 and the remaining keys are used in Building 2.

13. What is the probability that a key chosen at random is a gold key for Building 1?
14. What is the probability that a key chosen at random is a gold key for Building 2?
15. What is the probability that a key chosen at random is a key for Building 1?
16. What is the probability that a key chosen at random is a key for Building 2?
17. **Quality Control** At a factory, the percent of total output and the defective output rates for three machines are given in the following table.

	A	B	C
Percentage of total output	45%	30%	25%
Defective output rate	4%	5%	7%

What is the probability that an item selected at random is defective?

18. **Quality Control** Repeat Exercise 17 if the percent of total output for machines A, B, and C changes to 30%, 40%, and 30%, respectively.
19. **Weather Prediction** A local weather service predicts a 60% chance of rain. There is a 45% chance of high winds if it rains and a 15% chance of high winds if it does not rain. What is the probability that there will be high winds?
20. **Company Location** Of the companies in a city, 40% are considered to be large and 60% are considered to be small. The city is divided into three business districts: north, south, and east. Of the large companies, 26% are in the north, 32% are in the south, and 42% are in the east. Of the small companies, 38% are in the north, 44% are in the south, and 18% are in the east. What is the probability that a company selected at random is located in the northern part of the city? What is the probability that it is located in the eastern part of the city?

In Exercises 21 and 22, find $P(A | B)$.

21. $P(A) = \frac{2}{3}$, $P(A') = \frac{1}{3}$, $P(B | A) = \frac{1}{5}$, and $P(B | A') = \frac{1}{2}$
22. $P(A) = \frac{3}{8}$, $P(A') = \frac{5}{8}$, $P(B | A) = \frac{2}{5}$, and $P(B | A') = \frac{3}{5}$

23. **Reliability of Testing** One in every 200 people is infected by a virus. A test is used to determine whether a person is infected. If a person is infected, the test is positive 80% of the time, and if the person is not infected the test is still positive 5% of the time.
 - (a) If a person tests positive, what is the probability that the person is infected?
 - (b) If a person tests negative, what is the probability that the person is not infected?
24. **Reliability of Testing** Repeat Exercise 23, with the exception that 1 in every 100 people is infected by the virus.
25. **Television Coverage** A football game, a basketball game, and a hockey game are being played at the same time. There is a $\frac{1}{3}$ chance of the football game receiving local television coverage, a $\frac{1}{2}$ chance for the basketball game, and a $\frac{1}{6}$ chance for the hockey game. A sports announcer covers 60% of the football games, 30% of the basketball games, and 45% of the hockey games that are televised locally. If the announcer is covering the televised game, which sport is it most likely to be?
26. **Consumer Survey** In a consumer survey on toothpaste, 100 people try Brand A, 300 people try Brand B, and 200 people try Brand C. After the trial period, 65 people are satisfied with Brand A, 210 are satisfied with Brand B, and 150 people are satisfied with Brand C. If a person that tried one of these brands is chosen at random and is satisfied, what brand did that person most likely try?
27. **Quality Control** In Example 4, if a part that is chosen at random from the inventory is found to be nondefective, what is the probability that it came from Supplier 3?