Good Math Lesson Planning and Implementation

David Moursund

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"A person who dares to teach must never cease to learn." (Anonymous.)

"There's only one corner of the universe you can be certain of improving, and that's your own self." (Aldous Huxley; British writer, author of *Brave New World*; 1894–1963.)

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- Associate and then Full Professor, Department of Computer Science, University of Oregon. Served six years (1969–1975) as the first Head of the Computer Science Department.
- Full Professor in the College of Education at the University of Oregon for more than 20 years. Partially retired in 2002 and fully retired in 2007.
- In 1974, started the publication that eventually became *Learning and Leading with Technology*, the flagship periodical of the International Society for Technology in Education.
- In 1979, founded the International Society for Technology in Education. Headed this organization for 19 years.
- In 2008, founded the Oregon non-profit company Information Age Education.
- Author or co-author of more than 50 books and several hundred articles. Presenter of more than 200 professional talks and workshops.
- Major professor or co-major professor for 82 Ph.D. students—6 in Mathematics and 76 in Education.
- Click here for more information about David Moursund.

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"Mathematics consists of content and know-how. What is know-how in mathematics? The ability to solve problems." (George Polya; math researcher and educator; 1877–1985.)

"What science can there be more noble, more excellent, more useful for men, more admirably high and demonstrative, than this of mathematics?" (Benjamin Franklin; scientist, writer, one of the founding fathers of the United States; 1706–1790.)

Preface

"To achieve great things, two things are needed; a plan, and not quite enough time." (Leonard Bernstein; American conductor and composer; 1918–1990.)

Here are some of my observations about our educational system:

1. Our educational system has improved substantially over the past century.
2. Our educational system is struggling in effectively dealing with the current rapid pace of change in technology and other aspects of life in our world.
3. While today's students and the students of yesteryear share many characteristics, today's students are different in a number of ways that affect education.
4. The expectations being placed on teachers have substantially increased in recent years. Many teachers feel overworked, stressed, and under appreciated.
5. Our educational system has considerable room for improvement.

This book is about developing and implementing good math lesson plans. It is aimed at preservice and inservice teachers who teach math as part or all of their teaching assignment. The goal is to help improve math education.

The way you teach will be little affected by this book if you merely read it in a passive manner. You need to be actively engaged, reflecting on what I have written, and thinking about what it means to you. As an example, there are five statements given above. For each one, do you agree or disagree with it? Can you think of evident and personal experience that support or negate each statement? Can you add to the list? Do you talk about these types of topics with your fellow preservice or inservice teachers?

Aids to Teachers

Textbooks, teacher's manuals, and lesson plans are very good examples of aids to teachers. They represent the work of many learned and experienced teachers. Here are some other important aids to math teachers and their students:

1. Your students' innate human ability to learn to speak, comprehend, read, write, and think using natural languages (such as English, Spanish, Mandarin, Japanese). Your students can learn math.
2. The previous math knowledge, skills, experiences, and insights of your students. Math is a vertically structured discipline. Constructivism (students constructing new knowledge based on what they already know) plays a major role in a student's math learning processes. What you do in your teaching of math makes a huge difference to the future math teachers a student will work with.
3. Math manipulatives—be they physical (concrete) or virtual (computerized). Paper and pencil can be thought of as a math manipulative. Computers add a new dimension to the realm of math manipulatives.

4. Research in theories and practices of teaching and learning—including progress in brain science (cognitive neuroscience). This research helps build foundations for teaching and learning.
5. Computer-assisted learning and distance learning. These technologies extend the traditional aids to teaching and learning.
6. Calculators, computers, and computerized instruments that can solve or greatly help in solving many math-related problems and accomplishing many math-related tasks. This allows significant changes in the nature and extent of emphasis on some topics in the math curriculum.

Once again, I suggest that you pause and reflect about the list you have just read. What do the items in the list mean to you? How do they affect teaching and learning from your personal experience and points of view? What would you add to the list, and what would you delete from the list? From your personal point of view, what are the most important and least important items in the list? Good learning on your part is not memorizing the list and being able to regurgitate it on a test. It is developing a personally relevant level of understanding and being able to make use of that understanding in your teaching and learning.

Teachers and Their Lesson Plans

Humans are social creatures with tremendous innate ability to learn and to help each other learn. Every interaction you have with other people is a teaching and learning experience for you and the other people.

> A teacher-personalized math lesson plan is an extension of the teacher. It supplements and extends the human capabilities of a human teacher.

Lesson plans and lesson planning are an important component of teaching. This book is specifically intended for preservice teachers and for use in workshops for inservice teachers. People in each of these two groups will find material that will help them to become better teachers of math.

This book is not a compendium of math lesson plans. Indeed, it contains just a very few brief examples. You can find oodles of math lesson plans in books and on the Web. For access to a large number of math lesson plans that are available on the Web, see http://iae-pedia.org/Sources_of_Math_Lesson_Plans.

The accumulation of math lesson plans contributes to math education. However, if math education could be substantially improved by the accumulation and distribution of math lesson plans, it would be rapidly improving. There is something missing in this “formula.” What is missing are the human and the “theory into practice” components.

Each learner and each teacher is unique. As teachers and as learners we are not machines. Good lesson planning and implementation reflects the human capabilities, limitations, knowledge, and experience of both the teacher and the learners.

There are some aspects of teaching in which computers can out perform human teachers. We are living at a time in which computer-assisted learning and distance learning are gaining rapidly in capabilities, use, and importance. Good teachers and good teaching accommodate and make effective use of this major addition to the aids useful in teaching and learning. These newer aids, along with older aids, do not obviate the value of and need for good teachers and the need for good teachers. They do change the teacher's job. Remember, it is the teacher plus aids to the teacher that facilitate good teaching.

I think of a personalized math lesson plan as an extension of a human teacher. It supplements and extends the human capabilities of a human teacher. This is a unifying idea in this book.

Chapter 1: Introduction

"The longest journey begins with the first step." (Chinese proverb.)

"...we discovered that education is not something which the teacher does, but that it is a natural process which develops spontaneously in the human being. It is not acquired by listening to words, but in virtue of experiences in which the child acts on his environment. (Maria Montessori; Italian physician and educator, a noted humanitarian best known for the philosophy of education which bears her name; August 31, 1870–May 6, 1952.)

This book includes a number of instructional and inspirational quotations. Most are drawn from two Information Age Education sites:

- Math Education Quotations
- Quotations Collected by David Moursund

Math is one of the basics in education. It is expected that all students will move beyond the novice stage and develop math knowledge and skills needed for responsible adult citizenship. There are many aspects of this adulthood that directly or indirectly relate to and/or use math.

This book focuses on the design and implementation of good math lessons. However, this is not a book of sample math lesson plans. There are oodles of math lesson plans available on the Web and from other sources. By and large these lesson plans have three weaknesses:

1. They are not personalized to the individual strengths and weaknesses of the teacher, the teacher's students, and their culture.
2. They do not provide adequate insight into the math teaching and learning processes that help students grow in math maturity and develop long lasting math knowledge, skills, and habits of mind.
3. The person attempting to teach these lesson plans often has little personal involvement and ownership in the design and creation of the lesson plans.

If you have not already done so, spend a couple of minutes browsing the Table of Contents and reading the Preface. You will likely some topics that might interest you and your students. There is no need to read this book from cover to cover. Find a topic that interests you, and go directly to it.

Math Lesson Planning: It's Easy—Right?

This section is designed to get you involved in thinking about what might constitute a good math lesson plan. It is based on the written reflections of a fictitious preservice teacher.

On the first day of a math education course for preservice teachers, the following assignment was given:

> Each of you has learned a lot about education gained through your years of experience as a student and through the introductory education courses you have taken. Write a letter to yourself about your current insights into math lesson planning. The letter is not to turn in and share with the teacher. Rather, it is to be saved and reread at the end of the course.

Response from a Fictitious Student

Here is what a (hypothetical, fictitious, quite capable) preservice teacher wrote:

> My goal is to teach at the upper elementary school level. Math is not one of my favorite topics, and I have never been particularly good at it. However, I can do arithmetic okay, and I am quite sure I can handle the math in the upper elementary school grades.
>
> It seems to me that math will be one of the easier parts of my teaching assignment. As I think about it, I see five components to the task.
>
> **First,** I will receive a copy of the teacher's manual and the math textbooks. The school will also provide me with a syllabus that says what pages to cover, what I can omit, and what to emphasize for the state tests. I will count how many textbook pages are to be covered during the 180-day (36 weeks) school year. I will plan in terms of using four days a week to cover textbook pages, and one day a week for review, short quizzes and exams, snow days, fire drills, and so on. We will cover approximately the same number of pages during each of these page-coverage days—that is, the total number of pages to be covered divided by 144 (which is 4 days a week for 36 weeks).
>
> **Second,** in implementing the math content to be covered I will consistently use the following plan:

1. If homework has been assigned, collect the homework and deal with any questions the students have about it. Hand back the in-class seatwork papers handed in the previous day and answer questions about the previous day's material. This allows me to present a brief review of the previous day's content.
2. Spend about 10 to perhaps 15 minutes doing a "chalk and talk" presentation" (white board and projector presentation) of the new material. Remember to not get bogged down answering questions, as it is important to get through the material so students can then do their assigned math seatwork.
3. Give the students the worksheets (or, tell them the specific problems from the text) that they are to work on during the math period. Remind students what textbook pages we have just covered and suggest they refer to these pages if they have difficulties with the assigned activities. Make sure that some of the problems I assign are accompanied by answers in the back of the book or from some other source, so students can get some feedback on how well they are doing. Circulate around the room, answering questions that individual students have as they do the assignment.

4. Near the end of the period, ask the class if they have any questions over the material. Alternatively, use examples that individual students had trouble with while doing the assigned problems. If most students have made good progress on the assignment, collect the papers. If quite a few have not completed the assignment, assign the task as homework to be completed and turned in by the beginning of the math period the next day.

Third, give a short quiz or a longer test each week. Thus, in total, 36 class periods will be devoted to quizzes and exams. On *short quiz days*, the remainder of the math period will be used in playing on-computer and off-computer math-related games. Students who are having trouble with math and/or who have not completed required assignments will receive extra instruction and practice rather than participate in the fun activities.

During the month before the state math exam, part of each day will be spent reviewing for the test, taking practice tests, and learning test-taking strategies.

Fourth, I will make accommodations for diverse learners—especially students who are particularly slow in learning math and students who are particularly fast at learning math.

1. Students who are particularly good at math will be given some harder problems to work on after they have completed the assigned problems. Alternatively, they can volunteer to help the slower students in a peer-tutoring mode.
2. Students who do not complete the assignment during the available time will be required to continue working on the assignment during recess. I will supervise these students, because I assume the school will have a physical education teacher who handles recess activities.
3. I will seek parent volunteers to come into my class during math period to help students who need special attention and help in math. If I notice a student who has particularly difficulty, I will explore whether this student has one or more exceptionalities related to math learning and may need an IEP and perhaps a math tutor.

Fifth, I will dedicate part of my bulletin board display to math. I will post a math problem of the week, a math quote of the week, and a math joke or cartoon of the week. I will encourage my students to bring material that can be posted. For example, a student might bring an ad with a percentage discount, or an article with sports statistics.

> Lesson planning and lesson implementation are two sides of the same coin—they both must be well done if one wants to be an effective teacher.

What Do You Think of the Plan?

The plan writer obviously has taken some education courses and has been introduced to some of the important ideas. The plan reflects some careful thinking about the math teaching process, and it probably reflects some improvements on the way this preservice teacher was taught in elementary school.

What do you think are some of the strengths and weaknesses of this plan and the ensuing teaching/learning that it will facilitate? To help you in your analysis, here are a few thoughts that occurred to me (your author, David Moursund).

1. There is no mention of tying the math teaching and learning in with the current world the students live in —a world that includes multimedia, social networking, texting, and other routinely-used aids to communication and entertainment. For this preservice teacher, the textbook and teacher's manual define the math curriculum. Math is not tied in with any of the other subjects (such as science) the students are studying.
2. There is little emphasis on students learning to read the math book. Our overall math education system has a major weakness because many students do not learn how to read their math book or other math-related materials. By the time students get to algebra and higher-level math courses, many abhor word problems (story problems) and have great difficulty in dealing with such problems.
3. There is no emphasis on students learning to write and speak the language of math. There does not appear to be any general plan for students interacting with each other during math class, learning to learn from their peers, and helping their peers to learn. There is no mention of math journaling. Peer tutoring in mentioned only in terms of the math-gifted students—there is no mention of the whole class participating in paired learning.
4. There is no mention of math project-based learning or math problem-based learning.
5. Computers are mentioned as a possible aid to playing math-oriented games that the students might find enjoyable. However, there is no mention of students learning how to use calculators and computers as an aid to representing and solving math problems. Math modeling and computational thinking are not mentioned. Of course, it may be that the assigned textbook takes care of that. However, I have some doubts about this.
6. Physical and virtual (computer-based) math manipulatives are not mentioned. The same comment as in (5) above applies.
7. The use of computers as an aid to learning math is not mentioned. While the plan does not specify a strong emphasis on rote memory based on substantial drill and practice, there seems to be a hint of this in the plan.
8. While the bulletin board provides a little bit of insight into transfer of learning from the math instruction to other settings—both in school and outside of school—this really important aspect of math education receives very little emphasis.
9. The word *constructivism* and the underlying ideas of constructivism are not mentioned. A brief review of the previous day's math content is a long way from carefully exploring the prerequisite knowledge and skills assumed in a lesson and helping students who lack the essential prerequisites.
10. There is no mention of the teacher working to become a better math teacher, being a lifelong learner of math and math education, doing action research to benefit the teacher

and the teacher's professional colleagues, or taking a leadership role in improving math education in the school.

This list is easily extended. As you read and study subsequent chapters of this book, you will find items that seem important to you and that should be added to the list.

Types of Lesson Plans

Lesson plan usually refers to a single lesson, designed for one class period. However, it can also refer to a sequence of such plans designed for a unit of study. (Such a sequence may be called a unit plan.) In this book, lesson plan means a plan to facilitate one or more times of organized teaching and learning. Sometimes we will use the term "unit plan" to emphasize that we are talking about a sequence of one period lessons.

The following diagram illustrates that lesson plans can vary considerably in terms of their intended audience/use and the level of detail they contain.

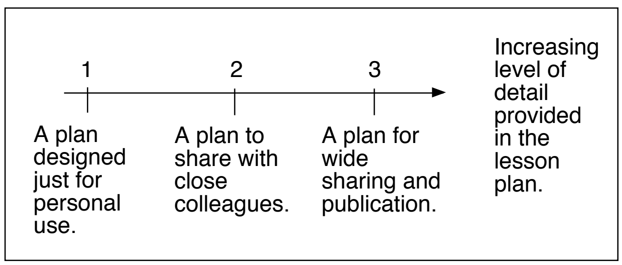


Figure 1.1. Types (levels) of lesson plans.

1. A personal lesson plan is a personal aid to memory that takes into consideration your expertise (teaching and subject area knowledge, skills, and experience). It's often quite short—sometimes just a brief list of topics to be covered or ideas to be discussed. For example:

   Give each student about 30 square tiles. The general goal is to explore forming connected geometric shapes that can be made from square tiles. Here, "connected" means that every tile in the geometric shape has at least one edge in common with another tile. Some of the shapes that can be formed have special names such as rectangle and square. Some are shaped like letters, such as an L. What letters can one make? What digits can one make? Figure out areas and perimeters of the connected figures. It is easy to see how to make different rectangles with areas 1, 2, 3, 4, 5, and so on. Can one make squares with areas 1, 2, 3, 4, 5, and so on? Why, or why not? Find examples of differently shaped rectangles that have the same area.

2. A collegial lesson plan is designed for a limited, special audience such as your colleagues, a substitute teacher, a supervisor such as a principal, or a person who is tutoring one of your students. It contains more detail than the first category. It is designed to communicate with people who are familiar with the school and curriculum of the lesson plan writer.

3. A (high quality) publishable lesson plan is much more detailed than a collegial lesson plan and is intended for use by a wide, diverse audience. It might be part of a teacher

education book or be posted on the Web. It is designed to communicate with people who have no specific knowledge of the lesson plan writer's school, school district, and state. It is especially useful to preservice teachers, to substitute teachers in unfamiliar situations, and to workshop presenters seeking to elicit in-depth discussion.

Electronic Digital Filing Cabinet

All teachers accumulate materials to use in their teaching. Some of the materials are physical objects that need to be stored on shelves or in filing cabinets. Physical math manipulatives provide a good example.

Some teaching and learning materials are best stored in a computer or a computerized storage device. Computer storage allows copies to be saved and used at home and school. It makes for easy sharing and updating. See, for example http://www.techteachers.com/mathematics.htm.

If you have not already done so, I strongly recommend that you build an electronic digital filing cabinet to assist you in your teaching and learning.

Here are two links to help you get started with building or improving your personal electronic digital filing cabinet.

- Digital Filing Cabinet/Overview.
- Math Education Digital Filing Cabinet

> Computer storage of your teaching materials allows copies to be saved and used at home and school, and easily shared.

Information and Communication Technology

Throughout recorded history, humans have worked to develop aids to representing and solving math-related problems. The abacus was such a successful aid that it is still used today, and bead frames are often used in math teaching and learning.

Information and communication technology (ICT) is important in math education. This book includes a focus on:

1. Roles of ICT in math curriculum content. ICT can solve or help solve a wide variety of math problems.
2. Roles of ICT in math instruction. Computer-assisted Learning (CAL) and Distance learning (DE) are now commonplace instructional delivery vehicles.
3. Roles of ICT in math assessment. ICT is now commonly used in math formative and summative assessment.

Final Remarks

Teaching is a very complex, challenging, and demanding profession. It is also a very rewarding profession.

> "...we discovered that education is not something which the teacher does, but that it is a natural process which develops spontaneously in the human being. It is not acquired by

listening to words, but in virtue of experiences in which the child acts on his environment. **The teacher's task is not to talk, but to prepare and arrange a series of motives for cultural activity in a special environment made for the child.**" (Maria Montessori; Italian physician and educator, a noted humanitarian best known for the philosophy of education which bears her name; August 31, 1870–May 6, 1952.) Bold added for emphasis.

> "The teacher's task is not to talk, but to prepare and arrange a series of motives for cultural activity in a special environment made for the child." Maria Montessori.

Math is a broad and deep discipline that has been developed by a very large number of practitioners and researchers over thousands of years. It is an integral component of our culture and our everyday lives. Good math teachers help to enrich the lives of their students.

End of Chapter Activities

This book is designed to be used in courses for preservice and inservice teachers. It can also be used in workshops and for self-study. Feedback is a key ingredient in learning. In any area you study, learning self-assessment (learning to provide feedback to yourself) is an important part of the learning process. Each chapter in this book ends with a section titled **End of Chapter Activities.** These can be used for reflection, self-assessment, to promote discussion in a course or workshop, or as assignments in a math education course.

1. Reflect on your insights into the fictitious naïve novice's planning for teaching math and on my comments on important ideas that might be missing. If you are using this book in a course or workshop setting, share your reflections with your fellow learners.
2. Math is a vertically structured discipline. New topics in a math class build on previously covered (covered—but not necessarily learned) topics. Reflect on how math prerequisites and math-learning constructivism are intertwined. Perhaps drawing on your own math education experiences, reflect on the difficulty of trying to build new math knowledge and skills on a weak foundation.
3. Reflect on your personal experiences with the use of calculators and computers in math education. What are your current thoughts on use of ICT and the proliferation of mobile connectivity and computing devices in math education?
4. Make up a question that you feel would be suitable for inclusion in this End of Chapter Activities section—and reflect on possible answers. (Here is an idea that you might want to think about. What do you think about asking students in a math class to pose problems, tasks, and other math related activities based on the math content they are studying? Problem posing is an often-overlooked content area in math teaching.)

Chapter 2: Overview of Lesson Planning

"Education is a human right with immense power to transform. On its foundation rest the cornerstones of freedom, democracy and sustainable human development." (Kofi Annan; Ghanaian diplomat, seventh secretary-general of the United Nations, winner of 2001 Nobel Peace Prize; 1938-.)

"In a completely rational society, the best of us would be teachers and the rest of us would have to settle for something less, because passing civilization along from one generation to the next ought to be the highest honor and the highest responsibility anyone could have." (Lee Iacocca, American industrialist; 1924-.)

Introduction

Proto humans and current humans have brains designed for learning about how to deal with complex problems, tasks, and challenges. We can plan ahead and consider possible consequences of proposed actions. We can visualize multiple possible future paths. Our distant ancestors survived in a world that included predators that far exceeded them in physical capabilities. Our ancestors exceeded these predators in ability to learn from each other, to work together, to develop tools to enhance their physical capabilities, and to pass on knowledge and skills from one generation to the next.

Each of us is innately a teacher and a learner. We routinely learn from each other and help each other learn. Some of us become "professional" teachers, steadily improving our skills in helping others to learn. You (a reader of this book) have spent years learning reading, writing, arithmetic, and various other disciplines currently deemed important in our society. You have repeatedly demonstrated that you are a skilled learner. Through reading this book, you are working to improve your skills as a future or current teacher.

There are many things that one can learn through imitation. In the hunter-gatherer era, children learned by imitation, by a "show and do" type of individual and small group tutoring, and by trial and error.

These same types of learning capabilities and approaches to learning still sufficed when the agricultural era began 10,000 to 11,000 years ago. Agriculture facilitated the development of permanent settlements and a variety of people who became specialists. Examples include pottery makers, basket weavers, flint knappers, and bow and arrow makers. We developed apprenticeship systems in which a learner studied under a master and might well spend many years becoming a specialist in a relatively narrow area. The use of apprenticeships is still an important part of our overall education system.

The governmental and business structures and problems that emerged during the first 5,000 years of the agricultural era eventually led to the development of reading, writing, and arithmetic. A child does not learn to read by observing an adult reading. Learning the 3Rs requires a different type of instruction.

In the early days of the 3Rs, some children received individual tutoring from a learned parent or professional tutor, and some came together in classrooms to be taught by a teacher. A very small percentage of the population received these types of instruction. The vast majority of people remained illiterate and innumerate.

In a worldwide basis a very high level of illiteracy and innumeracy remained the norm until recent times. Now, literacy and numeracy are considered a birthright. Sadly, there remain huge differences in the level and quality of formal education that children throughout the world are receiving. In some parts of the world many children grow up with no opportunity to go to school.

Literacy and numeracy are considered a birthright of today's children. Good teaching helps to ensure this birthright.

Teaching Versus Learning

I am a professional teacher. Via this book, I am attempting to help you learn. Learning is an individual and personal task. Learning takes place within your brain and the rest of your body. As you look at the "squiggles" that representing printing or writing, tough the raised dots of Braille printing, listen to music or audio books, view television or interact with a computer game, information flows into your brain. Your brain processes the information, interpreting it and understanding it based on long years of education and experience.

Your brain acquires and constructs new knowledge, connects it with existing knowledge, and stores it. Constructivism is a learning theory based on this process of constructing knowledge and understanding based on previous knowledge and understanding. Constructivism is an important learning theory in mathematics.

You can see the challenge that you and I face. I don't know what you already know and understand, and you want to learn some of what I know and understand. Every teacher and every student faces such challenges. They are particularly daunting in math and other vertically structured disciplines. At each grade level, a math course assumes that students know and understand the prerequisites—the content of previous math courses and earlier parts of the current math course. **Very often this is an incorrect assumption.** We know that students fail to learn or forget much of what they are taught in school.

Transfer of Learning

Probably you have heard the expressions "use it or lose it" applied to one's physical capabilities. The same expression holds for one's brain and learning. Students who have gained some specific math knowledge and skills tend to "lose it" over time through disuse. The learning becomes unavailable to transfer to (use in) situations they encounter in the future.

Students vary tremendously in the amount of math usage that they get at school—both in math classes and throughout the school day—and in their life outside of school. Thus, each student that you teach has unique preparation for constructing new math knowledge and skills.

I find it interesting to compare life outside of school and a student's life in taking a particular course. In school, we divide the content to be learned into disciplines. A student's school day might be divided into courses (lessons, units of study) that focus specifically on different disciplines such as the fine and performing arts, physical education, math, language arts, social science, and science. In middle school and high school these courses are typically taught by teachers who are not familiar with the detailed content of the day-to-day instruction students are receiving from other teachers.

Have you ever thought about the challenges students face as they try to transfer discipline-specific education to their other courses and to life outside of school? Math is a discipline that is useful in many other disciplines. But, transfer of learning is quite difficult. A good math lesson plan and a good math teacher help students transfer their current and new math knowledge and skills to other disciplines within school and to life outside of school.

Teaching for transfer of learning and student learning for transfer of learning are two of the major challenges in good math teaching.

What Defines a Discipline of Study?

When you teach math, you are helping students to learn the discipline of math. You know that the various disciplines taught in school are different. Think about similarities and differences between math and other disciplines as you read this following definition. A discipline is defined by:

- The types of problems, tasks, and activities it addresses. Its approaches to identifying, defining, and representing these problems, tasks, and activities.
- Its accumulated accomplishments, including results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on.
- Its history, culture, oral and written language, and other modes of communication—including notation, special vocabulary, gestures, and movement.
- Its methods of teaching, learning, assessment, and thinking. This includes what it does to preserve and sustain its work and pass it on to future generations.
- Its tools, methodologies, habits of mind, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results
- The knowledge and skills that separate and distinguish among people with varying levels of expertise in the discipline, such as:
    a. A novice—a rank beginner.
    b. A person who has a personally useful level of competence.
    c. A reasonable competent person, employable in the discipline.
    d. A local or regional expert.
    e. A national or world-class expert.

> Math teacher draw on their insights into and understanding of the math aspects of each of the six bulleted items. Some of these general insights into the discipline of mathematics can be (should be) integrated into the content and teaching of each math lesson.

Figure 2.1 provides a somewhat different representation of levels of expertise.

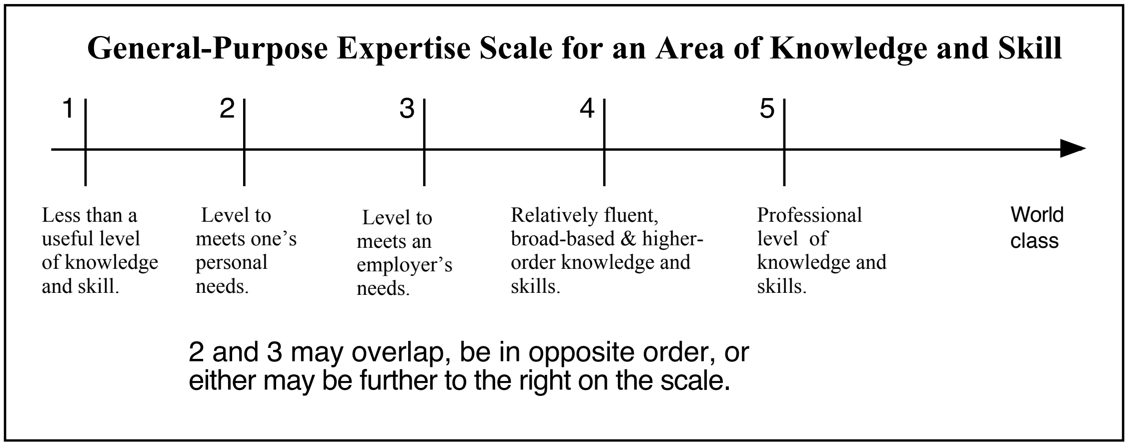


Figure 2.1. Scale of levels of expertise.

When you teach within a discipline, you represent that discipline. Part of your teaching task is the appropriate and adequate representation of the discipline. This means that you, as a math teacher, need to be able to identify and explain similarities and differences between math and the other disciplines your students have studied or are studying.

Remember, math has applications in many other disciplines. Good math teaching helps students learn some of the roles of math in other disciplines. This means a good math teacher needs to have insight into other disciplines and some of the roles of math in these disciplines. For example, you know that students use math in business and science. What do you—as a math teacher—do to help your students make a transfer of their math learning to science and business?

A General-purpose Lesson Plan

Lesson plans are a core theme of most preservice teacher education programs. Preservice teachers learn how to create them, how to critique lesson plans others have created, how to teach using a lesson plan, and how to assess the results of teaching from a particular lesson plan. By definition, a lesson plan is good to the degree it helps teachers teach well and students learn well.

The previous section provides a general definition of a discipline of study. This section provides a general-purpose outline for a lesson plan. It is applicable across a wide range of disciplines. Subsequent chapters of this book will provide you with help in adapting this general-purpose lesson plan to the math that you teach or are planning to teach.

Here is an 11-component outline for a general-purpose lesson plan. It is independent of any specific academic discipline or grade level.

1. Title and short summary. You may find it helpful to think of a title of a lesson plan as being like a section title in a book chapter, while the title of a multi-lesson unit plan is like a chapter title. The short summary can include information about how students will be empowered by learning the material in the lesson.
2. Intended audience and alignment with standards. Categorization by: subject or course area; grade level; general topic(s) within the discipline(s) being taught; length; and so on. A listing of the standards (state, or national) being addressed.
3. Prerequisites. It is difficult to state clearly the prerequisites for a particular lesson, and it is difficult to determine if students meet the prerequisites. A common approach consists of three parts:
   a. Very briefly summarize the general knowledge and skills of average students to be taught using the lesson.
   b. Indicate how you will do a quick assessment/review of the class's prerequisite knowledge and skills.
   c. Discuss what you will do for students who clearly lack the prerequisite knowledge and skills.
4. Accommodations. Special provisions needed for students with documented, relevant, significant differences from "the average" learners. The differences may be attitudinal, mental, or physical. The difference may so great that the lesson is beyond the student's capacity. Or, a student may have already mastered the lesson content and would be wasting time by participating in the lesson. In a typical third grade classroom there are apt to be some students who are at the first grade level and some at the fifth grade level in terms of knowledge and skills in a specific discipline.
5. Learning objectives. What do you want students to learn and be able to do as a consequence of the lesson? This is different than the question of what you will have students do during the lesson. The expression *measurable behavioral objective* is sometimes used in talking about learning objectives or learning goals. It is useful to think in terms of lower-order goals and higher-order goals, perhaps by using a modern version of Bloom's taxonomy.

   Quoting from Joyce and Weil (2000):

   > Most teaching episodes have both content and process objectives. The content objectives include the information, concepts, theories, ways of thinking, values, and other substance that the students can be expected to learn from the experience that results. The process objectives are the ways of learning—the conduct of the social and intellectual tasks that increase the power to learn. In the case of a model of teaching, the process objectives are those that enable the students to engage effectively in the tasks presented when the model is being used.
6. Materials and resources. These include written material for students to read, assignment sheets, worksheets, tools, equipment, CDs, DVDs, video tapes, physical environment, and so on. It may be necessary to begin the acquisition process well in advance of teaching a lesson, and it may be that some of the resources are available online.

7. Instructional plan. This is usually considered to be the heart of a lesson plan. It tells how to conduct the lesson.

   There are many different models of instruction. Some are more suited for teaching a particular discipline or a component of a particular discipline than others. The reader of a lesson plan may want to know the model of instruction it is using. Here are titles of a few examples of models of instruction (Joyce and Weil, 1996, page 401):

   - Cause and Effect (Inference, hypotheses, generalization)
   - Concept Attainment (Comprehension, Comparison, Discrimination and Recall)
   - Concept Development (Categorization)
   - Direct Instruction
   - Discussion (Essential questions)
   - Inquiry (Problem-solving)
   - Synectics (Use of group interaction to stimulate creative thought through analogical thinking) http://en.wikipedia.org/wiki/Synectics

8. Assessment options. A teacher needs to deal with three general categories of assessment:

   a. Formative assessment that provides timely feedback to students and the teacher as a lesson or unit of study progresses;

   b. Summative assessment that provides summary information about student learning (and, of course, effectiveness of teaching) at the end of a unit of study;

   c. Long-term residual impact assessment that occurs weeks, months, or even years later.

   In order to become efficient self-directed learners, students need to learn to do self-assessment and to provide formative assessment and perhaps summative assessment feedback to each other. A rubric, perhaps jointly developed by the teacher and students, can help students take an increasing responsibility for their own learning.

9. Extensions. These may be designed to create a longer or more intense lesson. For example, if the class is able to cover the material in a lesson much faster than expected, extensions may prove helpful. Extensions may also be useful in various parts of a lesson where the teacher (and class) decide they should spend more time on a skill or topic.

10. References. The reference list might include other materials of possible interest to people reading the lesson plan or to students who are being taught using the lesson plan. They should be references that are familiar to you and that you have personally judged appropriate.

11. Teacher reflection and lesson plan revision. This is to be done after teaching the lesson.

    a. What did you (the teacher) learn about the content, pedagogy, student readiness, and student learning? What additional content knowledge, general pedagogical knowledge and skills, and discipline-specific pedagogical knowledge and skills would help you to do better the next time you teach this lesson?

    b. Revise the lesson plan so that when you next use it and/or when you share it with your colleagues, it reflects your best current thinking.

> Professionals get better through reflective practice. View each lesson you prepare and teach as an opportunity to improve your lesson planning skills, your discipline content knowledge, your general pedagogical knowledge, your discipline-specific pedagogical knowledge, and your overall level of expertise as a teacher.

Final Remarks

Humans have innate learning and teaching abilities. Through study and practice you can greatly increase your levels of expertise in these two areas.

As you teach, keep in mind that each student you teach is and will continue to be both a learner and a teacher. Some of your students will become professional teachers. They will tend to teach in the way that they were taught. So, teach in a manner that you would like the children of the future to be taught!

End of Chapter Activities

1. Think back to the time when you were finishing high school or a GED. What were your strengths and weaknesses as a learner and as a teacher of the people you interacted with? Reflect on what you have done since then to become a better learner and a better teacher. Include in these reflections some of your current insights into what constitutes a good learner and a good teacher.
2. Select two quite different disciplines that are familiar to you. Using the six-part definition of a discipline given in this chapter, do a compare and contrast of these two disciplines.
3. Based on your experiences as a learner and/or on your experiences as a teacher, do a careful critique of the general-purpose lesson-planning guide given in this chapter. Look for strengths, weaknesses, and ways to improve the planning guide.
4. Reflect on the quality of teaching you have experienced as a student. Identify strengths and weaknesses you have witnessed and experienced.
5. Examine the list of models of instruction given in this chapter. Which of these have you experienced in your own math education? Which seem most effective to you in math teaching and learning?

Chapter 3: What is Mathematics?

> "A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas." (G. H. Hardy; English mathematician; 1877–1947.)
>
> … algebra is the language of mathematics, which itself is the language of the information age. The language of algebra is the Rosetta Stone of nature and the passport to advanced mathematics. It is the logical structure of algebra, not the solutions of its equations, that made algebra a central component of classical education." (Lynn A. Steen, American mathematician and math educator; 1941–.)

As an adult, you know a lot about math and you use your knowledge every day. Thus, if I ask you “What is mathematics?” you can give me an answer that fits with how you view and use math. You are empowered by your abilities to use math in dealing with money, time, distance, area, weight, statistical data, and other problem areas. Your insights into math and your uses of math provide you with a “reality check” as you help children to learn math through their informal and formal education.

However, the chances are that your answer to the *what is math* question is different than that of many other people. In terms of educating our children, it is helpful if we can have some agreement on what math is and what it is important for children to learn about math.

This chapter explores some answers to the *what is math* question. It provides background needed for developing and implementing good math lesson plans. As you read other people’s insights, reflect on how they fit in with your insights and on how they fit in with how you think math should be taught.

Algebra is the language of mathematics, which itself is the language of the information age.

A Short and Useful Answer to the What is Math Question

Here is one of my favorite quotes:

> “God created the natural numbers. All the rest is the work of man.” (Leopold Kronecker; German mathematician and logician; 1823–1891.)

Some knowledge and understanding of the amount or quantity represented by natural numbers 1, 2, and 3 is innate to humans. Very young infants have some innate understanding of small quantities. Studies of aboriginal tribes seems to have identified some that have words for only one and two, some have words for only one, two, and three, and some have move extensive vocabularies for quantity.

The current widely used human languages have words as well as math notation for 1, 2, 3, 4, 5, and so on. In this sense, the term *natural numbers* has come to mean the numbers 1, 2, 3, 4, 5, 6, 7, ….

It is easy to find patterns in the natural numbers. For example, there is the pattern that starting at 1, every second natural number (1, 3, 5, …) is odd. Starting at 2, every second natural number (2, 4, 6, …) is even. If we add two successive natural numbers, the sum is an odd natural number. If we multiply two successive natural numbers, the product is even. Some of the natural numbers have only two (exact) natural number divisors. We call these "prime" numbers.

With this background, we can summarize what mathematicians do. Here is my 3-step summary:

1. Working within the framework of the discipline of mathematics, a mathematician detects a possible pattern, describes the pattern, and conjectures that the pattern holds true in a carefully defined set of cases. (Example: I conjecture that the sum of two successive natural numbers is an odd natural number, and that this conjecture is true for every pair of successive natural numbers.)

2. The mathematician tests the conjecture in a number of cases, looking for counter examples and developing a "feeling" for why the conjecture might be correct. Examples: 1 + 2 = 3, an odd natural number; 2 + 3 = 5, an odd natural number; 3 + 4 = 7, an odd natural number. But, what about using larger natural numbers? Example: 17 + 18 = 35, an odd natural number; 1,644 + 1,645 = 3,289, an odd natural number.

   Hmm. If I start at the pair (1, 2), go to the pair (2, 3) and then to the pair (3, 4), I detect a pattern of the sums increasing by 2 each time. The sums I am generating for the sequence 3, 5, 7, 9, 11, 13, 15, … and that these are all odd numbers. A mathematician exploring a pattern looks for and is apt to detect other patterns. You may have noticed that the last digit in all of the odd natural numbers I have generated in my sequence of examples is one of the natural numbers 1, 3, 5, 7, 9. I wonder if there are any odd natural numbers that end in some other digit?

3. The mathematician attempts to develop a proof (an argument that is very convincing to both self and others) that the conjecture is correct. Here are some possible situations:

   a. The proof-finding effort may lead to finding an example in which the conjecture is incorrect.

   b. The situation being explored may consist of only a finite set of possibilities that need to be explored. A "proof" could then consist of exploring every possible one of this finite number and verify that the conjecture is correct in every case. Of course, if the number of possible cases is quite large, this could take a lot of time and effort. If a computer can explore the cases, this can save the human a lot of time and effort.

   c. The situation being explored may consist of an infinite number of possibilities. Mathematicians have developed techniques for dealing with such situations, such as in our example. In the early history of math, mathematicians had considerable difficulty dealing with situations involving infinity. You and your students may enjoy learning about Zeno's paradox (see http://www.mathacademy.com/pr/prime/articles/zeno_tort/).

d. Finally, the mathematician may not have the knowledge, skills, and persistence to resolve the situation. There are mathematical conjectures that have stumped mathematicians for hundreds of years.

Patterns are part of every discipline. As people explore patterns in a specific (non-math) discipline that interests them, they often find that math and the patterns it has studied prove useful. They may also find patterns that have not been incorporated into the discipline of math. If the patterns interest mathematicians, the study of these patterns will become part of the overall discipline of math. Through this process, math has become applicable in many disciplines of study and continues to become more applicable in the various disciplines that people study.

> In math, a proof is an argument that is very convincing to both self and others. It stands the scrutiny of others over time.

Math is a Human Endeavor

Long before we had written language and schools, people learned to count and make other simple uses of math.

Very young children have an innate ability to distinguish between the quantities one, two, and three. This, along with an innate ability to learn natural languages such as English and Spanish, provides a starting point for learning math.

From their early childhood caregivers, young children learn to count and how to determine the number of objects in a small set. Above that level of math understanding, knowledge, and skill, formal math education content begins to be introduced. For example, we have various numeral systems such as Roman numerals (I, II, III, IV, V), Hindu-Arabic numerals (1, 2, 3, 4, 5), Chinese numerals (一, 二, 三, 四, 五), and Arabic numerals (٤ ,٣ ,٢ ,١ ,٠). We have specially defined words and symbols for addition, subtraction, multiplication, division, and other operations on numbers. We have fractions and decimals. We have sub disciplines of math such as algebra, geometry, statistics, probability, and calculus. We have various systems for measuring distance area, and quantity, such as the metric system and the English system.

We have developed aids to solving math problems. For example, you probably own or have easy access to a calculator, computer, clock or wristwatch, and ruler. You routinely use these as aids to doing arithmetic calculations, to determine time of day, to measure length, and so on.

A car contains a speedometer and an odometer. Nowadays, it may contain a Global Positioning System (GPS) that can display a map and give you oral instructions as to when and where to turn to get to a specified destination.

Humans have developed many aids to doing and using math. If a readily available aid (such as a calculator or computer system) can solve or substantially help in solving a type of math problem being studied in school, what do we want students to learn about this type of problem? This is still a challenging and open question.

Thus, one way to answer the question "what is math" is to name some of its sub disciplines, name some of its tools, and name some of its achievements. However, this is of limited value in talking to and teaching students about math and helping them to develop personally useful understanding of the field. What mathematical knowledge and skills empowers children and students of various ages?

A Language for Thinking About, Representing, and Solving Problems

One good way to think about math is in terms of how it empowers people who seek to represent and solve a wide range of problems in different disciplines. Math is both a special language and a special approach to representing, thinking and reasoning about, and solving certain kinds of problems. Because there has been such a large amount of research in math over the years, there is a huge accumulation of how to solve a wide variety of math problems. If a problem in any discipline can be represented mathematically, this may be quite useful in solving the problem.

Here are four ideas that help to define goals of math education and empowering learners.

- Problem solving and proof lies at the very heart of mathematics. The discipline of math strives to represent and solve certain types of problems—and provide proof of the correctness of the results— in a manner so that future users of the results can build on them with confidence.
- Math fluency includes being able to read, write, speak, listen, think, and understand communication in the language of mathematics. This is somewhat akin to developing fluency in a natural language. The language of mathematics can be used to represent (model) problems from many different disciplines.
- Math maturity is being able to make effective and broad use of the math that one has studied. It is the ability to recognize, represent, clarify, and solve math-related problems using the math one has studied. Thus, a fifth grade student can have a high or low level of math maturity relative to math content that one expects a typical fifth grader to have learned.
- There is a huge amount of collected math knowledge, and it continues to grow. A student studying math—even one that spends a lifetime studying math—can learn only a small part of the accumulated math knowledge and skills. A good math education system represents substantial and careful thought as to how to most effectively spend a student's math learning time and effort.

Math fluency includes being able to read, write, speak, listen, and think, with understanding in the language of mathematics.

Learning Reading & Writing Versus Learning Math

Reading, writing, and arithmetic (more generally, math) are considered to be core components of a formal education. The average person has considerable innate ability to learn oral communication. It is easy to see a clear parallel between oral communication—speaking & listening—and the reading & writing combination.

A young child's level of skill in oral communication grows through being immersed in an oral communication environment and through years of practice. A child growing up in a linguistically rich home environment will grow in linguistic maturity faster than one growing up in a linguistically impoverished home environment. This increasing linguistic maturity is indicated by increasing correctness and effectiveness of oral communication—for example, being able to understand and carry out a complex set of instructions or to communicate a complex idea. Note also that if the child's home environment is bilingual or trilingual, the average child is quite capable of learning to communicate orally in two or three languages.

Our natural languages include words that facilitate counting and dealing with quantity. Thus, in societies that make substantial use of numbers, preschool children develop some counting and simple arithmetic skills, some number sense, and some understanding of the number line. A child's emerging and growing number sense and understanding of the number line are part of the child's growing level of math maturity.

Telling time using analog and digital clocks provides useful examples. A relatively young child can learn to read and say the numbers that represent a particular time on a digital clock. This is a far cry from understanding what the numbers mean. Time and the passage of a measured amount of time are complex ideas. As compared to a digital clock, an analogue clock is more challenging in learning to say the numbers that represent the time, but more user-friendly in helping a child understand lengths of time and the passage of time. Through years of practice and steady exposure to time and telling time, most people develop a level of "time maturity" that is expected of adults in our society.

People often refer to math as a language (Moursund, 2008b). Certainly math includes an extensive vocabulary and symbol set for oral and written communication. The language of mathematics facilitates very precise communication within the discipline of math.

The language of math is designed for a more precise level of understanding and communication than does a natural language. The lack of precision in an ordinary natural oral language communication typically does not lead to miscommunication. Small errors in grammar do not cause a problem. The same situation exists for writing in a natural language. Errors in spelling and punctuation may be annoying to readers, but do not usually lead to miscommunication. However, very small errors in using the language of math can lead to errors in answers produced by using math.

David Moursund's Answer to the "What is Math" Question

Many people have addressed the question, "What is mathematics?" See, for example, (Lewis, 1999) and the many publications of the National Council of Teachers of Mathematics. This answer to the "what is math" question comes from Moursund's book on computational thinking and math maturity (Moursund, 2006).

> Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems. The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making (Schoenfeld, 1992).

Notice the emphasis on thinking mathematically. One gains increased expertise in math by both learning more math and by getting better at thinking and problem solving using one's knowledge of math.

> Mathematics is built on a foundation that includes axiomatics, intuitionism, formalism, logic, application, and principles. Proof is pivotal to mathematics as reasoning whether it be applied, computational, statistical, or theoretical mathematics. The many branches of mathematics are not mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory. Hence, mathematical education should be centered on encouraging students to think for themselves: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof—the demand for succinct argument that from a logical foundation for the veracity of a claim (McLoughlin, 2002).

You can increase your level of expertise in math by learning more math and by getting better at thinking and problem solving using your math knowledge and skills.

Proof

The word *proof* comes up in most attempts to define mathematics. Of course, the idea of proof or proving something is not restricted just to mathematics. A trial lawyer attempts to prove his or her case. A person attempts to prove that another person is wrong in a particular situation. Researchers in science attempt to prove scientific theories.

Each discipline has its own ideas and standards about what constitutes a proof. Math proofs are designed to answer, once and for all, the correctness or incorrectness of a "mathematical" assertion. Suppose, for example, that I am exploring the sum of three consecutive integers. I see that 6 + 7 + 8 = 21, and 11 + 12 + 13 = 26. After looking at a lot of examples, I conjecture that if the first of the three consecutive integers is odd, then the sum is an even integer; if the first integer is even, then the sum is an odd integer.

Looking at lots of example, and not finding any counter examples may increase my confidence that my conjectures are correct. However, my failure to find a counter example does not constitute a proof. See if you can construct a convincing proof that my conjectures are correct. You might want to start by thinking about definitions of odd and even integers.

Then think about whether K-8 students, once they have encountered definitions of odd and even integers, might be able to develop convincing proofs. If the conjecture given above is too difficult for students at a particular age, how about considering the simpler conjecture that the sum of two consecutive integers is odd. A young child attacking this task might make use of small cubes. An even integer can be represented by two equal length rows of cubes. An odd integer can be represented by two rows, with one row having one more cube than the other.

Finally, be aware that there are lots of simple proof-type situations that can be constructed for use in the K-8 school setting. To give one more example, suppose that students have learned the mathematical word *mean*. You might then have them compute the mean of various sets of three consecutive integers, looking for a pattern. Quite likely some of the students will note that the answers they obtain are always the middle one of the three consecutive integers. Can they construct a convincing argument that this is always the case? What if one wants to find the mean of five consecutive integers?

When you present young students with such problems, you want to think carefully about what they may be learning. The examples given above might lead some students to think that the mean of a set of consecutive integers is an integer. With a little encouragement, some of your students might conjecture and then attempt to prove that "The mean of an odd number of consecutive integers is an integer, and the mean of an even number of consecutive integers is not an integer."

> Proof (careful arguments for correctness in solving problems and accomplishing tasks) can and should be built into the teaching of any math topic. Young students can learn to develop arguments that math work and thinking is correct or incorrect

Fluency and Proficiency

The terms fluency and proficiency are often used in talking about goals and expertise in mathematics. The following definition of math proficiency is quoted from Kilpatrick et al. (2001), a report written for the National Academy of Sciences. Mathematical proficiency, has five components, or strands:

- conceptual understanding—comprehension of mathematical concepts, operations, and relations;
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- strategic competence—ability to formulate, represent, and solve mathematical problems;

- adaptive reasoning—capacity for logical thought, reflection, explanation, and justification;
- productive disposition—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Warning! The mathematical proficiency bulleted list reflects many hundreds of hours of thinking by some of the world's leading math educators. Did you read it in a reflective manner? Did you work to construct your own meaning? What aspects of the presented ideas will you remember five minutes from now, a day from now, or a year from now?

For the most part, answers to the "what is math" question do not depend on specific areas of math content. The question and answers are part of math maturity. As you think about the mathematical proficiency bulleted list, you are working to increase an aspect of your math maturity that is very important to being a good teacher of math.

As you construct and/or make use of a math lesson plan, you can think about how it fits in with and contributes to increasing your students' mathematical proficiency. For example, compare having students work on a drill and practice page of arithmetic computations, versus students solving word problems, versus students creating word problems, versus students reading a science book and identifying math usage in science.

Gene Maier's Answer to the "What is Math" Question

The following is quoted from (Maier, 2000):

> … what mathematics is about. Making conjectures, seeking relationships, validating theories, searching for solutions, verifying results, communicating findings—in short, problem solving. To do mathematics is to solve problems.
>
> Problem solving--at least the phrase--has always been part of school math. Every math textbook series claims to emphasize it and every list of standards gives it special attention. Here in Oregon, the Department of Education's performance assessment in mathematics is a "problem-solving" test. Passing this test is a requirement for the Certificate of Initial Mastery—concocted as part of the Oregon Educational Act for the 21st Century in hopes of convincing the world that a tenth-grade Oregon education really means something.
>
> However, in contrast to the professional world of mathematics, school mathematics isn't synonymous with problem solving. In school, problem solving is likely to be considered just another topic, along with adding fractions, multiplying decimals, or finding perimeters. And so teachers attempt to teach it like any other topic …
>
> I suspect our students would be better problems solvers if we would quit treating problem solving as just another mathematics topic to be taught, but rather regard all mathematics as problem solving, and teach it accordingly. Most school math is ancient history--the mathematics that occurs in the curriculum are answers to mathematical questions that were posed years, or even centuries, ago. But these questions are new to our students.
>
> Mathematics doesn't have to be taught as a cut-and-dried, here's-how-you-do-it subject. It can be taught in a reflective, inquisitive mode. No matter what the topic, students' perceptions and suggestions can be explored, tested and refined. If every topic were

introduced as a problem to be investigated rather than a process to be mastered, there would be no need to treat problem solving as a separate topic with its own set of rules and procedures. Students' ability to solve problems would evolve naturally, hand in hand with their mathematical knowledge and sophistication.

> … our students would be better problems solvers if we would quit treating problem solving as just another mathematics topic to be taught, but rather regard all mathematics as problem solving, and teach it accordingly.

Final Remarks

Isaac Newton made a great many contributions in the fields of math and science. He said:

> "If I have seen further it is by standing on the shoulders of giants." (Isaac Newton; English mathematician & physicist; Letter to Robert Hooke, February 5, 1675; 1642–1727.)

It is the basic nature of math and science that researchers build on the work of previous researchers and strive to produce results that future researchers can build on. In both math and science, researchers work to identify (find, pose) problems that are fundamental in their disciplines and apt to be of lasting value and interest.

In physics and other sciences, math and math modeling are very important tools. Math is an interdisciplinary tool because it can be used in modeling (representing) problems in many different disciplines.

The advent of computers has brought up computer modeling—use of a combination of math and computers to do modeling and to carry out the necessary math and computing involved in a model. The combination of math and computers is such a powerful aid to problem solving that many disciplines now include math and computer modeling as a standard component of the discipline.

> Every child learning math is learning to stand on the shoulders of the people who developed the math.

End of Chapter Activities

1. This chapter contains the assertion, "math is a human endeavor." Many mathematicians think that it important to point out this "fact." Pick one or two other disciplines. To what extent do they also seem to be human endeavors? What makes one discipline more of a human endeavor than another?
2. This chapter illustrates that we have various numeral systems such as Roman numerals (I, II, III, IV, V), Hindu-Arabic numerals (1, 2, 3, 4, 5), Chinese numerals (一, 二, 三, 四, 五), and Arabic numerals (٤ ,٣ ,٢ ,١ ,٠). Likely you are familiar with at least two of these. Is one of them better for learning and doing arithmetic than the other? Why? What

are some math-learning values for a student to experience two different numeral systems and to do some compare/contrast thinking about advantages and disadvantages of each?

3. This chapter contains a section on Fluency and Proficiency. The section contains a list of five strands, beginning with *conceptual understanding.* Select a strand and reflect on what it means to you and how you can self-assess your understanding and competence in that strand area. Repeat the process for a different strand. Then do a compare and contrast between your insights and levels of strength in these two strands.
4. Select a grade level at which you currently teach math or have a goal of teaching math. Imagine being asked the "what is math" question by a student and by a parent of the student. Reflect on the answers you would provide. Pay special attention to what you want this student to learn about "what is math" through your teaching and what you expect the student has learned before coming into your math class.

Chapter 4: Math Maturity

"It isn't enough just to learn—one must learn how to learn, how to learn without classrooms, without teachers, without textbooks. Learn, in short, how to think and analyze and decide and discover and create. (Michael Bassis; President, Westminister College in Utah, USA.)

I like to divide math knowledge and skills into two (strongly overlapping) components—math content and math maturity.

M**athematically mature adults** have the math knowledge, skills, attitudes, perseverance, and experience to be responsible adult citizens in dealing with the types of math-related situations, problems, and tasks that occur in the societies and cultures in which they live.

To a great extent, math instruction focuses on math content. Math curriculum developers focus on what math content topics should be in the curriculum, in what order should they be taught, and how should students be assessed. Current state and national math assessment focus on math content and on solving problems based on this content. Indeed, to a large extent such testing is driving the curriculum.

However, there are many aspects of the overall discipline of math that are not assessed in state and national tests. (You might want to reread the definition of a discipline given in Chapter 2.) What do we want students to learn about math study skills, math problem posing, math habits of mind, and use of powerful computer aids to math problem solving? What do we want students to learn with "deep" understanding that will last for years or a lifetime? What do we want students to learn about transfer of math learning?

Here is a short summary of the goal of math maturity education. The goal is to produce mathematically mature adults.

> Mathematically mature adults have the math knowledge, skills, attitudes, perseverance, and experience to be responsible adult citizens in dealing with the types of math-related situations, problems, and tasks that occur in the societies and cultures in which they live. In addition, mathematically mature adults know when and how to ask for and make appropriate use of help from other people, from print materials and computerized information retrieval systems, and from tools such as calculators and computer systems. (Moursund and Albrecht 2011a.)

Defining Math Maturity

There is no widely agreed upon definition of math maturity. However, there is quite a bit of literature on the topic. My recent Web search of the expression *"math maturity" OR "mathematical maturity"* produced about 175,000 hits.

Quoting from the Wikipedia (http://en.wikipedia.org/wiki/Mathematical_maturity):

> Mathematical maturity is a loose term used by mathematicians that refers to a mixture of mathematical experience and insight that cannot be directly taught. Instead, it comes from repeated exposure to complex mathematical concepts.

I strongly disagree with the Wikipedia's assertion "cannot be directly taught." My position is that good math lesson plans and good math teaching include an explicit focus on increasing math maturity. I believe we can increase a student's level of math maturity through this approach to math education.

Many discussions of math maturity include a focus on understanding and creating proofs. Consider what this means for younger students. When a student solves a problem and then explains the steps involved in a manner that is convincing to others, the students has (in essence) created a proof. That is, problem solving and making proofs are two sides of the same coin.

Solving math problems and developing math proofs are two sides of the same coin.

George Polya was one of the leading mathematicians of the 20th century, and he wrote extensively about problem solving. *The Goals of Mathematical Education* (Polya, 1969) is a talk that he gave to a group of preservice and inservice elementary school math teachers. The talk focused on elementary school math.

> **To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics?** In the first place it means to be able to solve mathematical problems. For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems. [Bold added for emphasis.]

Notice the emphasis on problem solving. Polya also points out that doing a math exercise such as a long division is not the same as problem solving. Numbers can be added, subtracted, multiplied, and divided. Skill in carrying out these operations is useful in solving certain types of math problems. I like to think about this in terms of lower-order versus higher-order knowledge and skills. Arithmetic computation is lower-order, and problem solving is higher-order.

Sometimes I use analogy with writing. Spelling and penmanship are lower-order. Effective written communication is higher-order. In both math and writing, the instructional issue is how much emphasis to place on the lower-order versus how much emphasis to place on the higher-order. In writing education, there is now considerable agreement that lower-order and higher-order can and should be developed simultaneously. Math education has moved in this direction, but I think it lags the progress that has been made in writing education.

Continuing quoting from Polya (1969):

> However, … we wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. **Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions.** Some mathematics should enable a child at least to handle abstractions, to handle abstract structures. [Bold added for emphasis.]

The quoted material from Polya provides a good introduction to the topic of math maturity. We want students to learn to effectively deal with abstraction and thinking in math problem solving.

Math notation is part of the abstraction in math. Quoting from a course syllabus developed by Larry Denenberg (2003):

> Thirty percent of mathematical maturity is fearlessness in the face of symbols: the ability to read and understand notation, to introduce clear and useful notation when appropriate (and not otherwise!), and a general facility of expression in the terse—but crisp and exact—language that mathematicians use to communicate ideas. Mathematics, like English, relies on a common understanding of definitions and meanings. But in mathematics, definitions and meanings are much more often attached to symbols, not to words, although words are used as well. Furthermore, the definitions are much more precise and unambiguous, and are not nearly as susceptible to modification through usage. You will never see a mathematical discussion without the use of notation!

Math—including its language and notation—is inherently abstract. An increasing level of math maturity is shown by an increasing ability to deal with math-related abstractions.

Math Cognitive Development and Math Intelligence

Cognitive development and intelligence are closely related components of brain science. Students vary in their rate of general and math cognitive development and the level they reach. Students vary in their level of general and math intelligence.

Cognitive Development

Young children have an innate ability to deal with quantity at a concrete level. However, a number is an abstract concept. As a preservice or inservice teacher, you likely have encountered Piaget's theory of cognitive development. Jean Piaget's 4-stage theory of cognitive development has been somewhat modified over the years, but the underlying ideas have stood the test of time.

A person's brain slowly matures as it grows in size, learns to make effective use of its neurons, and develops more connectivity among the neurons. Physical maturity of a human brain is usually complete by age 25 or so. A physically mature brain has considerable ability to continue to learn and to adjust to new situations. Indeed, research suggests that:

… many cognitive functions including verbal memory, inductive reasoning, spatial reasoning, and verbal ability the peak does not occur until around age 53. And this is despite us losing 2 grams of our brain mass each year once we get past the age of 25. See http://www.brainhealthhacks.com/2011/01/12/what-could-account-for-our-middle-age-peak-in-cognitive-function-wiring/.

For many of a person's cognitive functions—including verbal memory, inductive reasoning, spatial reasoning, and verbal ability—the peak does not occur until around age 53.

Piaget's 4-state theory of cognitive development consists of:

Stage #	Name	Approximate Age Range
1	Sensorimotor	Birth to 2
2	Preoperational	Roughly ages 2 to 7
3	Concrete Operations	Ages 7 to 11 or 12, or higher
4	Formal Operations	Most people achieve their full brain development by about age 25 or so.

Figure 4.1. Piaget's 4-stages of cognitive development.

Here are important points.

1. Moving up the cognitive development scale corresponds to gaining an increasing ability to deal with abstraction.
2. The rate of cognitive development varies considerably with students.
3. Many people never fully achieve Formal Operations. Throughout their lives, they function at a Concrete Operations level, perhaps moving into the lower level of Formal Operations in some areas.
4. The level of cognitive development that a person reaches can vary from discipline to discipline and is dependent on cognitively challenging informal and formal education in the various disciplines. A student's level of cognitive development in math may be higher or lower than the student's level of cognitive development in some other discipline.
5. Many students have considerable difficulty in dealing with the abstraction inherent to the algebra coursework that now begins at about the 8th grade. Their overall level of cognitive development and their math level of cognitive development are not yet ready for the abstractness of algebra. A similar difficulty exists as many of these students move on to a 9th grade geometry course that includes a focus on proofs.

Innate Human Math Capabilities

People vary in their general intelligence. Moreover, some are far more gifted in mathematics than others. *The Math Gene* (Devlin, 1999) presents an argument that the ability to learn to speak and understand a natural language such as English is a very strong indication that one can learn math. The abstract language and notation (symbolism) in math provide the foundations that make math such a powerful aid to representing and solving problems,

Keith Devlin argues that a student's early development of math knowledge and skills is mostly dependent on informal and formal education coming from parents, teachers, television, games, and so on. See http://www.nctm.org/conferences/content.aspx?id=1270 for his opening keynote presentation at the 2004 NCTM Annual Conference.

Devlin's work helps us get at two major weaknesses in our current overall math education system. A great many parents were not particularly successful in learning math, and typically they do not provide a "rich" math environment for their children. A great many elementary school teachers are not particularly strong in math. These teachers tend to "cover" the math book and its related curriculum. As a consequence of the weak math levels of many parents and elementary school teachers, many young students are slowed in their journey toward math maturity that is consistent with their innate math potential.

Research indicates that several-week-old human babies have innate ability to recognize small quantities, such as noticing that there is a difference between two of something and three of that thing. A variety of other animals have somewhat similar innate sense of quantity. This initial number sense can be viewed as an initial (innate) level of math maturity.

Recent research supports the idea that a human brain also has some innate ability to deal with fractions. See http://www.medicalnewstoday.com/articles/145587.php. Quoting from the article:

> Although fractions are thought to be a difficult mathematical concept to learn, the adult brain encodes them automatically without conscious thought, according to new research in the April 8, 2009 issue of *The Journal of Neuroscience.* The study shows that cells in the intraparietal sulcus (IPS) and the prefrontal cortex—brain regions important for processing whole numbers—are tuned to respond to particular fractions. The findings suggest that adults have an intuitive understanding of fractions and may aid development of more effective teaching techniques.

In summary, informal and formal math education and math experiences build on an initial innate level of math maturity. The initial innate levels are demonstrated well before any oral language skills are developed.

Theory of Multiple Intelligences

Howard Gardner's theory of nine multiple intelligences includes logical/mathematical.

> **Logical/Mathematical Intelligence**: the capacity to understand the underlying principles of some kind of causal system, the way a scientist or a logician does; or to manipulate numbers, quantities, and operations, the way a mathematician does. (PBS, n.d.)

The Multiple Intelligences work of Howard Gardner (IAE-pedia, n.d) is part of a steadily growing collection of research on physical and mental diversity. Even identical twins have genetic differences. Through their life experiences, they develop experiential differences. No two people are identical.

Physical and mental differences among people present a challenge to a mass production type of educational system. The typical first grade teacher is faced by a collection of students who vary considerably both physically and mentally. The cognitive development of some of the students is a year or more behind the class average, while some other students are a year or more ahead of the class average. This presents a significant challenge to math teachers and our math education system—especially because math is a vertically structured discipline. Much of the math taught at any grade level is dependent on math covered in earlier grade levels.

> There are substantial differences between the brains of any two students—even the brains of identical twins. One key to improving math education is to appropriately address the issue of individual differences.

Dyscalculia

Dyscalculia affects a brain's math learning capabilities. Roughly speaking, dyscalculia is to learning arithmetic as dyslexia is to learning to read. The co-morbidity of these two learning disabilities is about 60% (Mills, 2011). Both dyslexia and dyscalculia strongly affect perhaps six to eight percent of the population. Both can be detected at an early age. Quoting from Mills:

> … about 60% of those diagnosed with either dyscalculia or dyslexia have the other condition as well. Further, it was already known that dyslexics had significant problems with math as well as with reading. Landerl's studies, however, showed clearly that the two conditions, dyscalculia and dyslexia, had completely different cognitive profiles and the symptoms were *additive* in the combined ("co-morbid") group. This suggested strongly that the two conditions affected different brain centers.

Dyslexia has been much more thoroughly studied than dyscalculia. Considerable progress has occurred in use of computers in an intervention designed to rewire a student's "reading" brain. The computer-based intervention is considerably faster and less expensive than the human specialist-based intervention. See, for example, the article about the Fast ForWord software at http://www.youtube.com/watch?v=mG8pZx9O-y8).

Cognitive neuroscience researchers have identified the specific brain area associated with dyscalculia and are developing interventions that help alleviate the problem Early research is summarized in (Attridge et al., n.d.). Quoting from that source:

> There is evidence that humans and some non-human animals have an innate Approximate Number System (ANS) that allows us to rapidly, but only approximately, represent numerosity. Children and adults appear to use these representations to compare, add and subtract non-symbolic quantities with above-chance accuracy.
>
> Furthermore, it has been suggested that the ANS may be the basis of formal mathematical ability in humans, and a relationship between individual differences in ANS acuity and mathematical ability in children has been demonstrated. Following from this hypothesis, interventions have been designed to strengthen the relationship between ANS representations and symbolic number representations in children, with the hope of improving their mathematics skills.

Mills (2011) presets more recent research on detection as well as success of computer-based interventions with students who have dyscalculia.

> Approximately 6 to 8 percent of students have dyscalculia. Early detection and strong interventions can make a huge difference in the math-related lives of such students.

Lower-order and Higher-order Cognition

In math, as in other disciplines, one can think of lower-order knowledge and skills versus higher-order knowledge and skills. Such discussions are founded on the work of Benjamin Bloom and what has come to be called Bloom's taxonomy. It is a six-point scale. Quoting from http://www.nwlink.com/~donclark/hrd/bloom.html, the six-point scale consists of:

1. Knowledge: Recall data or information.
2. Comprehension: Understand the meaning, translation, interpolation, and interpretation of instructions and problems. State a problem in one's own words.
3. Application: Use a concept in a new situation or unprompted use of an abstraction. Apply what was learned in the classroom into novel situations in the work place.
4. Analysis: Separate material or concepts into component parts so that its organizational structure may be understood. Distinguish between facts and inferences.
5. Synthesis: Build a structure or pattern from diverse elements. Put parts together to form a whole, with emphasis on creating a new meaning or structure.
6. Evaluation: Make judgments about the value of ideas or materials.

Bloom's taxonomy is designed to clearly differentiate between the lowest order (data and information, which he calls knowledge) and the three higher order levels (analysis, synthesis, evaluation) of human cognitive activity. The scale can be used in any subject area and for any grade level. In Bloom's research, he found that much (from his point of view, way too much) of the college level instruction was focusing on the three lower levels.

At any grade level in math teaching and learning, a student can be learning at both the lower-order and higher-order levels. A math student needs to learn some facts, but also needs to learn to think and solve challenging problems using the memorized facts. See http://www.pdfarticles.com/topic/blooms%20taxonomy%20questions%20math.html for articles containing specific applications of Bloom's taxonomy in math education.

Figure 4.2 is an unconventional way of thinking about lower-order and higher-order. It presents this topic from the point of view of an individual student working to gain increased expertise in a particular area. It posits that expertise comes from a combination of lower-order and higher-order components on the Bloom's taxonomy scale. It suggests that good teaching should focus on providing individual students an appropriate mixture of lower-order and higher-order components from the Bloom's taxonomy scale that are organized in a manner designed to increase the student's level of expertise.

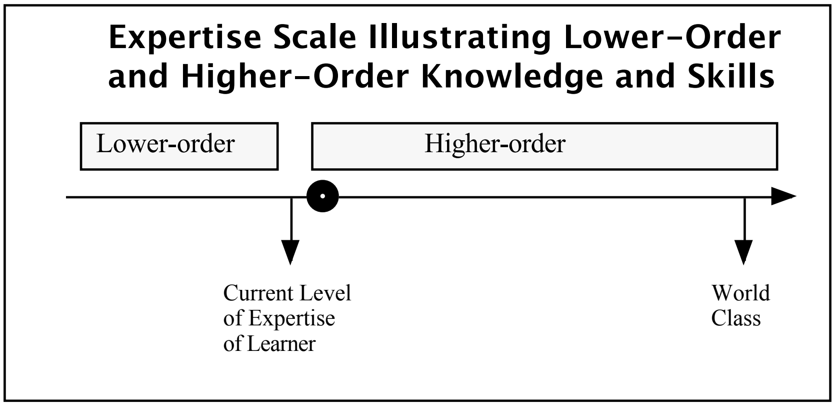


Figure 4.2. The black dot represents a student's instructional "sweet spot."

Expertise is a relative measure. One can think about expertise relative to the best in the school, the best in the school district, the best in the state, and so on. Figure 4.2 suggests that we consider a student's current level of expertise in a discipline, and then target the instruction at a slightly higher level. From this point of view, lower-order means below a student's current level of expertise and higher-order means above a student's current level of expertise.

> In math learning, a student's instructional "sweet spot" is highly dependent on the student's current level of math content knowledge and skills, and the student's level of math maturity. This helps to support arguments for individualization in math education.

Some Components of Math Maturity

This section provides an extensive list of some generally agreed on components of math maturity. As you read this section, think about your own level of math maturity in each of the areas listed. Also, think about how you might teach for increased math maturity in each of the component areas, and how you might assess student progress in each of these areas.

1. Communication

Mathematics is a language. We want students to learn to communicate mathematics and math ideas orally and in writing using standard notation, vocabulary, and acceptable style. We want them to learn to think in the language.

Consider a situation in which a person is using oral language, gestures, written languages, pictures, and diagrams, to communicate a math problem or math related problem to another person. The idea is to communicate the problem carefully and fully, so that the receiver of the communication can then bring his or her math knowledge and skills to bear in attempting to solve the problem. This type of communication requires a high level of precision on the parts of the two participants. An increasing level of math maturity is indicated by an increasing ability to communicate—read, write, speak, listen—in and think in the language of math.

2. Learn to learn math and help others learn math

Learning to learn math. Learning math means learning with an appropriate combination of memorization and understanding. Some key ideas include constructivism, metacognition, and reflective thinking. Learning to learn math includes learning to make effective use of the various aids to learning that are available, such as teachers, peers, books, and computers. It also includes leaning to make effective use of one's overall learning knowledge and skills, and one's specific math learning knowledge and skills. An increasing level of math maturity is evidenced by increasing ability to be a self-sufficient intrinsically motivated learner who learns math with understanding. Increasing math maturity is indicated by an increasing ability to work with people having varying levels of math knowledge and skills, and to help them learn math.

3. Generalize from a specific example to a broad concept

Categories of problems. Mathematicians often start from a specific example of a problem, and then go on to represent, define, and solve a broad category of closely related problems. For example, one might start with a concrete example of a problem involving a specific equilateral triangle, and develop results that solve this type of problem for every equilateral triangle. Increasing math maturity is indicated by getting better at identifying a general class of problems from a specific example, in solving the general class of problems, and in making use of a solution to a general class or problems to solve specific instances of the problem.

4. Transfer of learning

Transfer of math learning. One aspect of learning math is to learn a variety of strategies (algorithmic and heuristic) that are useful in attacking a broad range of math problems, and to learn to develop such strategies on one's own. Another aspect of learning math is to learn to think and reason mathematically and to apply this skill to the math problems one encounters. Increasing math maturity is indicated by getting better at transferring or applying one's math knowledge and skills into other areas of math and into math related areas and problems in disciplines outside of mathematics. Progress is shown by increasingly being able to apply one's math knowledge and skills to challenging math-related problems and problem situations that one has not previously encountered.

5. Multiple, varied representations

Concrete and more abstract representations. Children begin their leaning of math well before they reach the "concrete operations" phase of cognitive development. This early math learning is rooted in verbal, tactile, and visual representations of specific concrete objects and events. Increasing math maturity is evidenced in increasing ability to deal with generalizations and less concrete examples. For example, there is a difference between working with three blue toy cars and two red toy cars that one has physically sitting before one's eyes, and doing the same thing with pictures in a book or pictures of cars in one's mind's eye. At a much higher math level, increasing math maturity includes getting better at moving back and forth between the visual (e.g., graphs, geometric representations) and the analytical e.g., (equations, functions) math representations.

6. Math problem solving and proofs

Problem solving and proofs. Solving math problems and proving math theorems lie at the very heart of mathematics. Increasing math maturity is evidenced by progress in being able to provide solid evidence (informal and formal arguments) of the correctness of one's efforts in solving math problems and making proofs. This is a specific type of communication in the language of math. Before students encounter formal math proof processes, math proof is often stated as "Show your work and check your answers." This means to present written explanations and arguments that support the assertion that your work is correct. Increasing math maturity is indicated by increasing ability to read, understand, and create proofs.

7. Math-related word problems

Word problems. Represent (model) verbal and written problems in any discipline as mathematical problems. Recognize when a word problem might make effective use of math in attempts to solve the problem. Increasing math maturity is indicated by increasing knowledge and skills in representing word problems using the language of mathematics, solving the resulting math problem, translating the results back into the language and context of the original word problem, and checking for accuracy and mindfulness of the math results in light of the context and meaning of the original word problem.

8. Math is a human endeavor

A human endeavor. Math is more than just solving math problems and making math proofs. Our accumulated math knowledge represents considerable human creativity over thousands of years. Math is part of our culture. Math is fun. Math is part of the games we create and play. Math is part of the beauty of our world. (Stephen Brown, 1996.) Increasing math maturity is evidenced by increased understanding or and participation in these various aspects of the overall discipline of mathematics.

Some related topics include mathematizing, thinking like a mathematician, and (math) problem posing (Stephen Brown 1997). An increasing ability to mathematize (see the math in a problem situation; pose math problems) is an indicator of an increasing level of math maturity.

9. Math content

Specific math content. As you think about the math maturity components listed above, notice that few specific math content topics are mentioned. One needs to know some math in order to be able to demonstrate increasing math maturity. But, increasing levels of math maturity are only loosely connected to learning a prescribed list of carefully specified and widely agreed on collection of math content topics.

An indication of an increasing level of math maturity is a student's active engagement in improving his or her math education and prowess in using this math education. This effort need not be focused on just the curriculum content being offered by one's school. Indeed the ability and interest to explore self-selected math-related ideas that happen to seem interesting is a good indicator of an increasing level of math maturity.

Great effort has gone into the development of various curricula and various approaches to learning math. There is a reasonable amount of agreement as to math topics that students should learn something about in elementary school, secondary school, and in undergraduate programs of study. However, it is depth of understanding that is the key idea.

Thus, part of one's increasing math maturity is increasing breadth and depth of mathematical knowledge and skills. However, because of the great breadth and depth of the discipline of math, a mere listing of topics one has studied is not a good indicator of math maturity. A growing list of topics that one has learned with understanding is an indicator of increasing math maturity.

10. Mathematical intuition

Math intuition. Herbert Simon, a Nobel Prize winning polymath, defined intuition as "frozen analysis." He noted that in a discipline that one studies and practices extensively, a subconscious type of intuition is developed through the careful and repeated analyses one has done. This intuition may well be able to quickly detect an error that one has made in math thinking and math problem solving, very quickly decide a way to attack a particular type of problem, or provide a "feeling" for the possible correctness of a conjecture. As one's knowledge of and experience in using math grows, one's math intuition grows. This is a good indicator of increasing math maturity.

As an example, consider a student paper that indicates 5 + 8 = 40. At a subconscious level your brain might say, "something is wrong." It might next tell you, "the number 40 is way too large." Your experience and math teaching intuition might tell you, "perhaps the student multiplied instead of added." Through grading lots of student papers, you have developed some math intuition that makes you into a faster paper grader.

We see such intuition in other areas, such as in chess. An accomplished chess player can glance at a board position and have a "feeling" for the threats and opportunities that the position represents.

For a deeper view of math intuition, see Henri Poincaré (1905). Quoting the first paragraph:

> It is impossible to study the works of the great mathematicians, or even those of the lesser, without noticing and distinguishing two opposite tendencies, or rather two entirely different kinds of minds. The one sort are above all preoccupied with logic; to read their works, one is tempted to believe they have advanced only step by step, after the manner of a Vauban who pushes on his trenches against the place besieged, leaving nothing to chance. **The other sort are guided by intuition and at the first stroke make quick but sometimes precarious conquests, like bold cavalrymen of the advance guard.** [Bold added for emphasis.]

> Math intuition grows through the extensive study of, use of, and reflection about math.

11. Math habits of mind

. Quoting from Costa and Kallick (n.d.):

> A "Habit of Mind" means having a disposition toward behaving intelligently when confronted with problems, the answers to which are not immediately known. When humans experience dichotomies, are confused by dilemmas, or come face to face with uncertainties, our most effective actions require drawing forth certain patterns of intellectual behavior. When we draw upon these intellectual resources, the results that are produced through [their use] are more powerful, of higher quality and greater significance than if we fail to employ those patterns of intellectual behaviors.

The book Moursund and Albrecht (2011a) contains a math education analysis of the 16 habits of mind developed by Costa and Kallick. Here is an example from that book:

# 2 Habit of Mind: Managing impulsivity.	Application to Math Education. This habit of mind is applicable both in interacting with other people and in carrying out tasks such as problem solving.
Think before you act, and consider the consequences of your actions before taking the actions. Remain calm, thoughtful, and deliberate. Don't be driven by a need for instant gratification; with practice, one can learn to control this impulse.	In math problem solving, one has a goal in mind. Learn to mentally consider various approaches to achieving the goal. Learn to analyze whether the steps one is taking or considering taking will actually contribute to ward achieving the goal. Students who are driven by the need for instant gratification seem to have trouble in their math studies when they reach algebra. See http://i-a-e.org/newsletters/IAE-Newsletter-2009-24.html.)

12. Computers and other math tools

Information and communication technology. All of the components listed above need to take into consideration the various tools that have been developed to aid in representing and solving math problems and problems in which math can be a useful aid to their solution.

Calculators and computers are powerful examples of such tools. These tools are useful both in representing and solving math problems and also in learning math. Moreover, computational mathematics is now one of the major subdivisions of the overall field of mathematics. Thus, increasing levels of math maturity are indicated by increasing knowledge and skills in making effective use of Information and Communication Technology (ICT) as an aid to representing and solving math problems, as an aid to learning math, and as an aid to math-related communication (Moursund, 2007).

The Marshmallow Test of Self Control

This section discusses some very interesting (and amusing) research about delayed gratification (Lehrer, 5/18/09). It relates closely to the topic of impulsivity mentioned in (11) above.

Youngsters are tested on whether they can delay eating a marshmallow (or some other "treat") in order to get two of the treats 15 minutes later. Only about 1/3 of the four-year old US children in the original research and 1/3 of the 4–6 year old Colombian children in research on children in that country were able to delay for 15 minutes.

Follow-up research on the US children 15 years later indicated that all who were able to delay their gratification for 15 minutes had been quite successful as students and in other parts of their lives.

Quoting from Lehrer article:

> Once Mischel began analyzing the results, he noticed that low delayers, the children who rang the bell quickly, seemed more likely to have behavioral problems, both in school and at home. They got lower S.A.T. scores. They struggled in stressful situations, often had trouble paying attention, and found it difficult to maintain friendships. The child who could wait fifteen minutes had an S.A.T. score that was, on average, two hundred and ten points higher than that of the kid who could wait only thirty seconds.

…

Angela Lee Duckworth, an assistant professor of psychology at the University of Pennsylvania] first grew interested in the subject after working as a high-school math teacher. "For the most part, it was an incredibly frustrating experience," she says. **"I gradually became convinced that trying to teach a teen-ager algebra when they don't have self-control is a pretty futile exercise."** And so, at the age of thirty-two, Duckworth decided to become a psychologist. One of her main research projects looked at the relationship between self-control and grade-point average. She found that the ability to delay gratification—eighth graders were given a choice between a dollar right away or two dollars the following week—was a far better predictor of academic performance than I.Q. **She said that her study shows that "intelligence is really important, but it's still not as important as self-control.** [Bold added for emphasis.]

Self-control is a habit of mind that can be learned through instruction and practice.

Final Remarks

Students face many challenges as they strive to learn math. The nature and extent of these challenges vary considerably from student to student. Thus, a "one size fits all" approach to math education fails to meet the needs of a great many students.

Math education has two major and strongly overlapping threads: math content and math maturity. Our math education system places far more emphasis on math content than it does on math maturity. I strongly believe that our math education system can be improved through developing a better balance between these two overlapping aspects of math education.

Good math lesson plans are designed to help students gain in math content knowledge and skills, and in math maturity.

End of Chapter Activities

1. Suppose you are talking to a student and the student says, "I'm too dumb to learn math." Formulate a good response to this "dumb" assertion.
2. Among the Howard Gardner's list of nine multiple intelligences, which do you feel is your strongest area of intelligence and which do you feel is your weakest? (See the list at http://skyview.vansd.org/lschmidt/Projects/The%20Nine%20Types%20of%20Intelligence.htm.) Reflect on how a teacher can help a student grow in logical/mathematical intelligence.
3. Select three or four of the components of math maturity discussed in this chapter. For each, analyze your strengths and weaknesses. Among these components, determine the component in which you have the greatest math maturity and the component in which you have the weakest math maturity.

4. Reflect on your math-oriented habits of mind. Identify some of your strengths and some of your weaknesses. Reflect on ways you could improve some of your weaker math habits of mind.
5. Reflect on the extent to which need for instant gratification is a “driver” in you life. What do you do to rein in this driver?

Chapter 5: Problem Solving

"Each problem that I solved became a rule which served afterwards to solve other problems." (René Descartes, French philosopher, mathematician, scientist, and writer; 1596–1650.)

"Learning to solve problems is the principal reason for studying mathematics." (National Council of Teachers of Mathematics.)

Math problem solving lies at the very heart of learning math and learning to use math. Good math lesson plans—and good math teaching—focus on math problem solving. René Descartes' quote given above is applicable in math and in other disciplines. It captures a key idea about transfer of your learning among disciplines and to new problems you will encounter in the future.

What is a Problem?

Each academic discipline includes an emphasis on representing and solving problems. In my professional career, I have found it to be an interesting and enlightening challenge to write a definition of "problem" that cuts across all disciplines. Here is my definition of problem. As you read it, reflect on how well it fits for different disciplines you have studied.

You (personally) have a problem if the following four conditions are satisfied:

1. You have a clearly defined given initial situation.
2. You have a clearly defined goal (a desired end situation). Some writers talk about having multiple goals in a problem. However, such a multiple goal situation can be broken down into a number of single-goal problems.
3. You have a clearly defined set of resources that may be applicable in helping you move from the given initial situation to the desired goal situation. These typically include some of your time, knowledge, skills, and physical capabilities, and cognitive capabilities. Some of the resources available for your use might include friends, money, paper and pencil, books, and ICT. There may be specified limitations on resources, such as rules, regulations, guidelines, and time limits for what you are allowed to do in attempting to solve a particular problem.
4. You have some ownership—you are committed to using some of your own resources, such as your knowledge, skills, time, and energy, to achieve the desired final goal.

The "resources" in the third part of the definition do not tell you how to solve a problem. Rather, they are used as aids to problem solving. In many problem-solving situations, Information and Communication Technology (including the Internet, Web, calculators, and computers) and computerized tools are resources. These resources have grown more powerful over the years. That is one reason why it is becoming more and more common to thoroughly integrate the use of computers in problem solving into the basic fabric of academic courses.

The fourth part of the definition of a problem is particularly important. Unless you have ownership—through an appropriate combination of intrinsic and extrinsic motivation—you do not have a problem. Motivation, especially intrinsic motivation, is a huge topic in its own right.

Many students find they have little or no personal interest—little intrinsic motivation, little ownership—in the types of math problems and many other topics that they encounter in school. Thus teachers, parents, grading systems, and other extrinsic motivation factors are used to motivate and/or coerce students into learning (Vockell, 2006).

This is not a good situation in math and other disciplines that students study. Perhaps the challenge is more severe in math education than in most other disciplines. Starting at about the fourth grade, students must deal with a great deal of delayed gratification. They are steered through a curriculum that prepares them for the next year's curriculum that then prepares them for the next year's curriculum. They eventually are steered through algebra, geometry, and more algebra. Most of the content they are learning is not immediately useful or used in their everyday lives or in the other courses they are taking.

Ownership of a problem to be solved, intrinsic motivation, and self-control are all important aspects of math education.

Math educators often try to distinguish between a problem and an exercise. Consider a work sheet containing 25 two-digit multiplications to be performed. I consider this overall activity and the individual computations to be exercises. This is a drill and practice activity that is designed to help increase a student's speed and accuracy in using an arithmetic algorithm. Note that a calculator can do such computations, and that a computer can do millions of these per second without any errors. In my opinion, such computational, algorithm-based activities of this sort are not worthy of being called problems.

Often the statement or representation of a (proposed) problem lacks precision in one or more parts of the definition. Indeed, a part may be missing. Then we say that the problem is ill defined or not clearly defined, and we call it a problem situation rather than a problem. One of the challenges of problem solving is to start with an ill-defined problem situation and extract from it a well-defined (clearly defined) problem that is relevant to the original problem situation.

What is Problem Solving?

I use a very broad definition of problem solving. It is useful in many different disciplines. Problem solving includes dealing with:

- Question situations: recognizing, posing, clarifying, and answering questions.
- Problem situations: recognizing, posing, clarifying, and then solving problems.
- Task situations: recognizing, posing, clarifying, and accomplishing tasks.
- Decision situations: recognizing, posing, clarifying, and making good decisions.
- Using higher-order critical, creative, wise, and foresightful thinking to do all of the above. Often the results are shared, demonstrated, or used as a product, performance, or presentation.

Transfer of Learning

Learning for transfer of learning is one of the most important ideas in education. The goal is to gain and retain knowledge and skills that can be used in the wide variety of problem-solving situations one encounters after the initial learning.

The low-road/high-road theory of learning has proven quite useful in designing curriculum and instruction (Perkins and Solomon, 1992). In low-road transfer, one learns something to a high level of automaticity, somewhat in a stimulus/response manner. In essence, this is a description of a near transfer type of situation.

For an example, consider the situation of students learning the single digit addition or multiplication facts. This might be done via work sheets, flash cards, computer drill and practice, games, competitions, and so on. Most students require a lot of drill and practice over an extended period, along with subsequent frequent use of the memorized facts.

Moreover, many students find that they have difficulty transferring their math knowledge and skills from the math class school environment to other subject areas in school and to applications outside of school. One of the difficulties is recognizing when to make use of the memorized number facts. In school, the computational tasks are clearly stated; outside of school, this is often not the case.

High-road transfer for improving problem solving is based on learning some general-purpose strategies and learning how to apply these strategies in a reflective manner. A strategy is a general plan. The strategy of breaking a complex problem into two or more less complex problems provides an excellent example. This strategy is often called the divide and conquer strategy

If you can solve the less complex problems you do so, and then put the results together to solve the original problem. If you have created a sub problem that you cannot solve, you may want to try breaking it into two or more still less complex problems.

Math education research suggests that a typical student has a very small repertoire of math problem-solving strategies. Often a student applies these in a haphazard manner—with little though as to how they might apply in a particular problem. Thus, helping a student to learn a new strategy and to develop some math maturity in its use can be a quite worthy math education goal.

> Many students find that transfer of their math learning is a particular challenge. An increasing level of skill and versatility in transfer of math learning is a good indicator of an increasing level of math maturity.

Polya's 6-Step Strategy for Problem Solving

George Polya (1887-1985) was a great mathematician and teacher who wrote extensively about problem solving. Polya's 6-step problem-solving strategy was designed for use in math problem solving. The following version of this strategy has been modified to be applicable in many problem-solving domains. All students can benefit from learning and understanding this strategy and practicing its use for a wide range of problems. This problem-solving strategy assumes that a student has some personal commitment to (ownership in) solving the problem.

1. Understand the problem. Among other things, this includes working toward having a clearly defined problem. You need an initial understanding of the Givens, Resources, and Goal. This requires knowledge of the domain(s) of the problem, which could well be interdisciplinary.
2. Determine a plan of action. This is a thinking activity. What strategies will you apply? What resources will you use, how will you use them, in what order will you use them? Are the resources adequate to the task?
3. Think carefully about possible consequences of carrying out your plan of action. Place major emphasis on trying to anticipate undesirable outcomes. What new problems will be created? You may decide to stop working on the problem or return to step 1 because of this thinking.
4. Carry out your plan of action in a reflective, thoughtful manner. This thinking may lead you to the conclusion that you need to return to one of the earlier steps. Note that reflective thinking leads to increased expertise.
5. Check to see if the desired goal has been achieved by carrying out your plan of action. Then do one of the following:
    a. If the problem has been solved, go to step 6.
    b. If the problem has not been solved and you are willing to devote more time and energy to it, make use of the knowledge and experience you have gained as you return to step 1 or step 2.
    c. Make a decision to stop working on the problem. This might be a temporary or a permanent decision. Keep in mind that the problem you are working on may not be solvable, or it may be beyond your current capabilities and resources.
6. Do a careful analysis of the steps you have carried out and the results you have achieved to see if you have created new, additional problems that need to be addressed. Reflect on what you have learned by solving the problem. Think about how your increased knowledge and skills can be used in other problem-solving situations. Work to increase your reflective intelligence!

It is very important to understand that there is a huge difference between memorizing Polya's 6-step strategy and in developing the knowledge, skills, and math maturity that make routine use of this strategy when it might be useful. Moreover, a strategy is different than a solution process. Having and using a strategy does not guarantee success in solving a problem.

> Polya's six-step strategy for approaching math problems provides useful guidance. With practice, its use becomes a valuable math habit of mind.

There are other broadly applicable math problem-solving strategies. Look it up (in a textbook, in a library, on the Web) can be an effective strategy. You know that computers can solve a very wide range of math problems and provide help in solving a still wider range of problems. Thus, the look it up on the Web strategy may result in finding a site that can solve the problem.

Suppose that a calculator and/or a computer can solve a type of math problem that you are covering in a math curriculum. What do you want your students to learn about solving this type of math problem? In developing a lesson plan for this type of math problem, how do you appropriately and effectively cover the issue of “by hand” versus “ICT-assisted” solving of math problems? A good math lesson plan moves well beyond the goal of just being able to get a particular type of problem solved!

Natural Language and the Language of Math

I know how to read, write, speak, listen, and think both in the natural language called English and in the language of mathematics. Students have varying capabilities in these endeavors. It is a challenge to learn to effectively communicate in the language of mathematics (Moursund, 2008b).

> Math is a very important component of the sciences and a number of other disciplines. Many students find that weakness in their math knowledge and skills restricts their progress in such other disciplines.

The natural languages that humans use to communicate with each other contain many words that are used to communicate math ideas. Thus, in English we have number words such as one, two, and three. Math uses these same words but also the "abstract" symbols 1, 2, and 3. In English we have the words plus, minus, times, and equals. Math uses these words and the abstract symbols +, -, x, and =.

However, there is a difference between the languages. In a natural language, a word may have several different meanings, while a math symbol typically has a very precisely given single meaning. “Thus, when we say, "All men are created equal" the word "equal" does not have the same meaning as the math symbol =. One of the challenges in learning math is to learn to effectively deal with and communicate precisely using the words and symbols of math.

Many students come to fear or abhor math word problems. Word problems are often called story problems. In this book, the term "word problem" is taken to mean any problem that can be communicated via oral language, sign language, written language, gestures, drawings and paintings, video and audio recordings, and so on. Often word problems are communicated in a combination of a natural language and the language of math.

We use number words and numbers in representing and dealing with age, time, distance, length, area, volume, money, and so on. In all of these situations, the number words are accompanied with some unit of measure. A student is 14 years old. A grocery store item costs $5.28. Carpet is sold by the square yard—and a certain amount of money per square yard. Distance is measured in units such as cm or inch, meter or yard, kilometer or mile, and so on.

In natural language, we often make use of somewhat confusing and quite approximate measures. When asked, "How far is it from Eugene, Oregon to Portland, Oregon?" a person might respond, "About two hours." That answer is a driving time for traveling by car. However, if two bicyclists were talking, the answer given might be "most of a day."

Figure 5.1 illustrates typical steps followed in solving a math word problem. This six-step process draws heavily from Polya's six-step strategy.

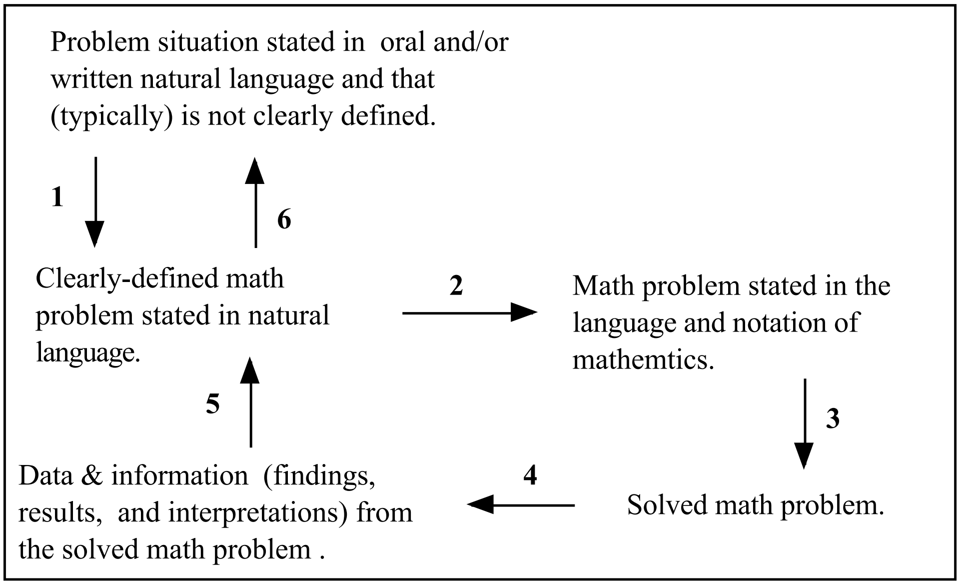


Figure 5.1. Six-step process used in solving a math word problem.

I find it quite instructive to examine the six steps in Figure 5.1. Here are a few of my insights:

1. Step 1 is a thinking step. It requires knowledge about the problem situation under consideration. If the problem is coming from a specific discipline, then the problem solver needs to know some things about that discipline. The problem solver also needs to understand differences between an ill-defined problem and a well-defined problem in that discipline.
2. Step 2 is a math-modeling step. The goal is to create a well-defined and clearly stated math problem that represents the problem posed in step 1. Math modeling is a very important component of a good math education. Nowadays, it is common to do computer modeling—to develop a computer model of a problem and then use a computer to help explore and solve it. See Computational Thinking at Moursund (2006) and http://iae-pedia.org/Computational_Thinking.
3. Step 3 is a math problem-solving step. It requires math knowledge, skills, and thinking. This is the step most emphasized in math courses. Keep in mind that for many problems, this step can be carried out by a calculator or computer. In many other problems, calculators and computers are useful aids.

4. Step 4 can be thought of as a math-unmodeling step. The results from solving the math problem are translated back into the vocabulary and ideas of the problem resulting from carrying out step 1.
5. Step 5 analyzes whether the well-defined problem posed in step 1 has been solved.
6. Step 6 analyzes whether the original problem or problem situation has been resolved. It is also a time for reflecting about the whole problem-solving task and what has been learned in solving the problem.

Problem solving is a thinking activity. Through practice and with the help of aids such as computers, some aspects of math problem solving can become routine and mechanical.

An overriding challenge in math education is to steadily improve the math thinking abilities of students.

The "Story" in Story Problems

Math education places considerable emphasis on solving word problems (Moursund, 2011). Many students find math word problems to be quite difficult because of the dual challenge of needing to translate the problem statement into a math problem and then to solve the math problem. Such students find it is easier to learn to solve "pure" math problems than it is to learn to translate word problems into math problems.

The "story" in the statement of the problem may help or hinder a student. Here are two word problems.

1. I am thinking of a number. If I double the number and then subtract 13, the result is 77. What is the number I am thinking of?
2. If I double how old I am and subtract 13, I get my mother's age. My mother is 77. How old am I?

These are essentially the same problem. The second problem provides some "real-world" information (context) about people and ages of people. It allows problem solvers to draw on their knowledge that a child is younger than his or her mother. This fact can be used as an aid to verifying the possible correctness of a proposed solution to the problem.

In this simple-minded example, the context of real-world referents is of only modest use. For example, a negative age answer makes no sense, and everybody knows that a child is younger than his or her mother. When math is used to help solve a real world problem, the context of the problem often provides information about the reasonableness of a proposed answer.

Thus, one aspect of a "good" word problem is that it provides context and referents that the problem solver can combine with his or her real world knowledge as an aid to checking the reasonableness of a proposed answer.

This is a very important idea in math education. Consider a clerk in a store keying in the number and/or price for an item that is not in the scanner system. It is very easy to make keying errors.

An important part of an answer to this error detection question lies in drawing on knowledge about the problem situation. For example, perhaps I am buying some fruit and I remember distinctly that I picked out the cheapest of the various varieties of apples from the various apple bins. I didn't want to spend more than about $4 or $5 for apples. I recall that some apples cost much more than others. I glance at the display being generated as the clerk processes each item, and I see that I have bought about $12 worth of apples. No! Surely I didn't buy that many. Something has gone wrong! Perhaps I took the apples from the wrong bin. Perhaps there was a keying error, a scanner error, a weighing error, or wrong pricing information has been entered into the store's computer.

A quite similar situation occurs when I use a calculator to do a calculation. Perhaps I am working on my family budget, on income tax, or on figuring out student grades at the end of a term. It is very easy to make a keying error. But, I have some knowledge about what constitutes reasonable answers. I often catch keying errors (including results from keys that stick, accidental depression of two keys at once, accidentally depressing a key twice when I don't mean to, and so on). If I am mentally alert while doing the calculations, I will detect an "A" student suddenly becomes a "C" students or when I unexpectedly seem to owe the U.S. Internal Revenue Service a lot of extra money. Of course, it is possible that my calculations are correct. But surely, I need to do them over again to help make sure I have not made a keying error.

This type of "common sense" in math problem solving is a very important aspect of math maturity. It applies both to good word problems (that include useful "real world" context and referents) and to math problems that are clearly stated in the language of math. In the latter case, mathematicians talk about developing an intuitive feel for the correctness of an answer. Through years of experience in doing math they develop an ability to sense that a result they have produced is probably not correct —perhaps due to an error in computation, symbol manipulation, or logic.

> In math and in other disciplines, "common sense" tends to be uncommon. It is improved by learning to understand the meaning(s) of the problems one is trying to solve, and then using that meaning in solving the problem and checking the results that are obtained.

Word problems are an important component of a good math curriculum. However, the "goodness" of a particular word problem or set of word problems must be judged in terms of the contribution made to achieving the overall math education goals that guide the curriculum, instruction, and assessment.

Problem-based Learning

Problem-based learning can be part of the curriculum in any academic discipline. Business students do case studies and medical students solve problems involving analyzing diagnostic data and prescribing treatments. The objective is to present students with problems that are appropriately challenging—that stretch their problem-solving skills and make use of the discipline's content that they are studying.

Quoting from the ERIC Digest article Roh (2003):

> Problem-Based Learning (PBL) describes a learning environment where problems drive the learning. That is, learning begins with a problem to be solved, and the problem is posed is such a way that students need to gain new knowledge before they can solve the problem. Rather than seeking a single correct answer, students interpret the problem, gather needed information, identify possible solutions, evaluate options, and present conclusions.

The four 4s is an example of math problem-based learning. The goal is to combine four 4s in various ways in order to make as many different integers as possible. The "combine" rules are that one can use addition, subtraction, multiplication, division, and parentheses. Thus: (4 + 4)/(4+ 4) = 1; 4/4 + 4/4 = 2; and (4 + 4 + 4)/4 = 3. For a more complex version of the problem, change the rules to also allow concatenation (thus, 44/44 = 1 and 444/4 = 111), exponentiation, or other types of operations.

This math problem and its variations are widely used in math education. It illustrates that a math problem may have more than one solution. It illustrates the need for very careful definition of a problem. It is a problem that can engage individual students or a team of students over an extended period of time. All students in a class can be assigned the same problem. Some students are likely to produce more answers than others.

There are many variations of the problem, both in the base number (for example, how about using four 3s) and the allowable operations. A Web search of *four fours math problem* produces many thousands of hits.

See http://www2.edc.org/mathproblems/favorites.asp for some more examples. See http://pbln.imsa.edu/model/intro/ for ideas on how to teach math using problem-based learning.

Problem-Based Learning begins with a problem to be solved. The problem needs to be sufficiently challenging so that students gain in math problem-solving skills as they attempt to solve the problem.

Problem-based learning can be done in a collaborative learning environment. Teams of two or more students work together on a problem. There has been substantial research on use of collaborative learning in various academic disciplines. David W Johnson and Roger T Johnson have long been leaders in this area of research. Quoting from Johnson and Johnson (n.d.):

> In the mid-1960s, cooperative learning was relatively unknown and largely ignored by educators. Elementary, secondary, and university teaching was dominated by competitive and individualistic learning. Cultural resistance to cooperative learning was based on social Darwinism, with its premise that students must be taught to survive in a “dog-eat-dog” world, and the myth of “rugged individualism” underlying the use of individualistic learning. While competition dominated educational thought, it was being challenged by individualistic learning largely based on B. F. Skinner’s work on programmed learning and behavioral modification. Educational practices and thought, however, have changed. cooperative learning is now an accepted and often the preferred instructional procedure at all levels of education. Cooperative learning is presently used

> in schools and universities in every part of the world, in every subject area, and with every age student. It is difficult to find a text on instructional methods, a teacher's journal, or instructional materials that do not discuss cooperative learning. Materials on cooperative learning have been translated into dozens of languages. Cooperative learning is now an accepted and highly recommended instructional procedure.

The Johnsons have identified five important elements of cooperation across multiple classroom models:

- Positive interdependence
- Individual accountability
- Structures that promote face-to-face interaction
- Social skills
- Group processing

In math education, all of these can be viewed aspects or areas of math maturity.

Final Remarks

Problem solving draws on both lower-order and higher-order cognitive skills. It is a routine part of everyday life, and it is a part of every academic discipline. Math is a challenge to students for many reasons, including:

1. It is required of every student—whether they want to study this area or not.
2. It is a vertically structured discipline, with success at any grade level being dependent on having gained math knowledge and skills at previous grade levels.
3. It is applicable across the curriculum—and thus presents students with a challenge of needing to know both math and the discipline area of the problems they are attempting to solve.
4. There are powerful tools (such as computers) that can aid in math problem solving, but it can be a challenge to learn to effectively use these tools.

End of Chapter Activities

1. Consider a word problem that contains the information that Pat is 14 years old. Does this mean that it is Pat's birthday? Are we talking about 14 years later to the nearest hour, the nearest minute, the nearest second since Pat was born? What if Pat was born on February 29 of a leap year? Give some other examples of ambiguity in a natural language that can lead to ill-defined math word problems. Some of the problems discussed at http://iae-pedia.org/Math_word_problems_divorced_from_reality may provide you with help in this exploration.
2. Compare and contrast Polya's six-step strategy with the strategy given in Figure 5.1. Are they just two different representations of the same thing, or are there important differences in their purposes and uses?
3. Reflect on your capabilities, experiences, and attitudes in dealing with math word problems.

4. Make up your own definitions of “math exercise” and “math problem.” Spend some time browsing the Web looking for challenging math problems (not exercises) that are suitable for use at a grade level you teach or are planning to teach. Reflect on criteria that make the problems good (or, not good) for use in problem-based learning.

Chapter 6: Lesson Plan Implementation?

"One good teacher out weighs a ton of books." (Chinese proverb.)

A teacher implements a lesson plan—right? A good lesson plan should incorporate and reflect the teacher's knowledge and skills that are relevant to implementing the plan. It should play to a teacher's strengths.

But what if a plan is to give students a large block of time to work on math worksheets? Who is actually implementing the plan—the teacher, the worksheets, or the students?

What if students have learned to read a math book and have learned to learn math by reading math. A substantial part of a math lesson might be students reading their math books. Is it the book and the students who are implementing the plan?

What if the plan is to have students do project-based learning, and a large block of class time is allocated to students working together on a project. Who is implementing the plan?

What if the plan is for students to work in pairs in a paired tutoring mode? Who is implementing the plan?

What if the plan is to have students make use of distance learning materials? What if the plan is for students to make use of computer-assisted learning materials? Is it the media facilities and computers that are implementing the plan?

By now, you get my point. Planning and implementation are two different things, although they are closely related. We are used to teacher-centered implementation. This chapter explores implementations that are not teacher centered. In some, the role of the teacher is far different than a "stand and deliver" mode. In others, the teacher may play a minor or no role in implementation.

Curriculum, Instruction, and Assessment

You have external and internal senses that bring information from outside or inside your body into your brain. Your brain and the rest of your body process information, make decisions, and implement the decisions.

Your brain and the rest of your body learn through such processes. Your brain can consciously help guide some of these learning processes. Other learning occurs at a level below your consciousness.

So, what does this mean in terms of the teaching and learning of math? Our informal and formal education system provides math-related learning environments. A math course provides environments that are designed to help you learn specific math curriculum that has decided upon and developed by math education experts. We can analyze a math course in terms of curriculum content, instructional processes (pedagogy), and assessment.

Over thousands of years, the accumulated knowledge of the human race has grown and grown and grown. Books, reading and writing, and a wide range of information storage and retrieval systems have made this possible. What content should students learn?

Many countries have developed what they consider to be a good curriculum for their residents. In the United States, we now have a Common Core State Standards (CCSS) that has been adopted by most states. (See http://www.corestandards.org/the-standards/mathematics.)

Over the years we have developed aids to instructional delivery. Chalk and a chalkboard, film projectors, audio recorders and playback devices, overhead projectors, video projectors, marker pens and a white board, computers, and smart boards have each proven useful in delivering content to a student's senses. As each new delivery system comes along, people do research designed to provide evidence that the new aids are in some ways more effective than the old.

Over the years we have also developed aids to formative assessment, summative assessment, and long-term residual impact assessment. These forms of assessment each provide feedback to various groups, such as students, parents, teachers, politicians, curriculum developers, assessment developers, and educational researchers.

When I think about the content, pedagogy, and assessment paragraphs given above, I see the potential for steadily increasing success in education. We just continue incremental improvements in each of the three major components. As they say, however, sometimes the devil is in the details. I believe that our overall educational system continues to improve—but the demands placed on the system and the needs of the students continue to change and to grow. Incremental improvement may not be a successful approach in dealing with the changes going on in technology, our society, and the world.

Curriculum content, instructional processes, and assessment (formative, summative, and residual impact) can each be incrementally improved over time.

Curriculum Content

Think about some aspects of Information and Communication Technology that are in wide use. The Web is an interactive digitized global library. It accumulates lots of information and makes it available to Web users. A telephone can be used to communicate over long distance and thus exchange information with people at a distance. A computer can solve or help solve a wide variety of problems. Computerized instruments and tools, such as a Global Positioning System and industrial robots, can be used to highly automate the solving of some complex problems.

These technology-based tools raise questions about curriculum content. What do we want students to learn, and how well do we want students to learn it? These are very difficult questions. They are made more difficult by our rapid growth in the totality of accumulated knowledge, by rapid improvements in tools that aid our mental and physical capabilities, and by continuing research on teaching and learning.

Historically—and, continuing still today—a significant part of curriculum content and learning has involved rote memorization. With practice, the average human is quite good at this.

A person can memorize many songs, poems, and lists with little insight into their messages and meanings. A person can memorize how to spell many words—and even to give examples of sentences using the words—without getting better at writing. A student can memorize the math rule "invert and multiply" with little insight as to when this rule is applicable and why it works.

How should curriculum content reflect the steadily improving capabilities of technological aids to solving problems and accomplishing tasks? For example:

1. What data and information should one memorize versus what should one become skilled in "looking up?" I certainly envy the information retrieval knowledge and skills of a good research librarian.
2. What algorithmic and procedural tasks should one memorize and become highly skilled at versus what should one learn to do making use of computers and computerized aids?

These are hard questions, and answers change over time as better aids are developed and needs of people change. In brief summary, my belief is that we should spend less school time teaching students by-hand and by-memory methods for doing things that computers and other tools can do very well. Many people agree with this and try to move our curriculum towards higher-order thinking and creative problem solving.

In every academic discipline, a good lesson plan reflects careful thought about the curriculum content. Teachers and students need to understand how the curriculum reflects a compromise among many competing stakeholder groups. Teachers and students are two of the stakeholder groups. Because they are "where the rubber meets the road," they can and should play an important role in choice of curriculum content.

Curriculum content also needs to reflect the individual needs and differences of the students being educated. These individual differences may be physical and cognitive, but they are can also be cultural, family-based, and community-based. We have no trouble accepting profound individual differences situation in special education, but we have trouble accepting it in general education and in talented & gifted education.

> Curriculum content needs to reflect the individual needs and differences of the students being educated. It also needs to reflect the steadily improving aids to problem solving.

Instructional Delivery

The locus of control in instructional delivery varies widely with different methods of instruction and aids to instruction. Consider schools and schooling before the development of printing presses. A teacher was the source of the content to be taught, and instructional processes necessarily centered on the teacher. Teachers presented information, students memorized information, students copied the information (perhaps onto their slates), and then students did oral and written drill and practice activities.

Books

The development of relatively inexpensive paper and printed books slowly changed this situation. I have found it interesting to learn about the McGuffey Readers. A small collection of books dominated elementary school educational in the United States for a great many years. (See http://mcguffeyreaders.com/1836_original.htm.) Quoting from the Wikipedia:

> McGuffey Readers were a series of graded primers that were widely used as textbooks in American schools from the mid-19th century to the mid-20th century, and are still used today in some private schools and in homeschooling.
>
> It is estimated that at least 120 million copies of McGuffey's Readers were sold between 1836 and 1960, placing its sales in a category with the Bible and Webster's Dictionary.

The McGuffey Readers removed some of the instruction delivery burden from teachers. Paper and quill pen (and, eventually, pencils and other types of pens) were integrated into the instructional delivery and learning processes. If you find this type of history interesting, you might want to look at the early history of the school library. For example, see http://www.americanlibrariesmagazine.org/ask-ala-librarian/first-school-library. Some history about aids to math teaching and learning is given at http://www.pballew.net/mathbooks/.)

Interestingly, the McGuffey Readers did not cover arithmetic or other math. For the most part, students continued to learn math via what I call an "oral tradition" method of teaching. Math books contained formulas and procedures to be memorized and problems to be solved.

However, a number of classroom aids to the teaching, learning, and doing of math were incorporated into regular classroom use. Quoting from http://www.bbc.co.uk/schools/primaryhistory/victorian_britain/victorian_schools/:

> What was a Victorian [British, mid 1800s] classroom like?
>
> There were maps and perhaps pictures on the wall. There would be a globe for geography lessons, and an abacus to help with sums. Children sat in rows and the teacher sat at a desk facing the class. At the start of the Victorian age, most teachers were men, but later many women trained as teachers.
>
> Children wrote on slates with chalk. They wiped the slate clean, by spitting on it and rubbing with their coat sleeve or their finger! Slates could be used over and over. For writing on paper, children used a pen with a metal nib, dipped into an ink well.

See a combined slate and bead frame in Figure 6.1.

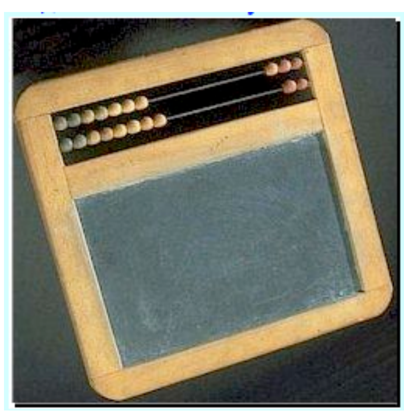


Figure 6.1. Bead frame and slate tablet.

Quoting from http://www.pballew.net/mathbooks:

> In William B. Fowle's report to the trustees of the Boston Monitorial School in 1825, he says: "Every child in school is furnished with a slate and pencil, which are considered a part of the furniture of the school."

Here is a personal story:

> My mother was born in 1907. She did a master's degree in mathematics at Pembroke, the women's college in Brown University (Rhode Island). She liked to tell the story of a professor who came to class each day with his long-used, yellowed, math notes. He would read the notes to the students (reading each sentence twice) at a pace that allowed the students to copy what he was reading.
>
> Before her graduate work, my mother taught in a secondary school. Once I asked if her students had a good supply of paper. Her somewhat defensive reply was that the school was located near a paper manufacturing company and her school was able to get "mill ends"—the remnants from huge, long rolls of paper—so access to paper was not a problem.

Likely you are familiar with the ideas of "reading across the curriculum" and that our schools have a strong focus on students learning to read well enough by the end of the third grade so that reading to learn can be an important component of instructional delivery. Year-by-year, reading becomes a more and more important component of instructional delivery. In many disciplines, by about the seventh grade it is the dominant delivery system.

However, this is not the case for math instruction. Here are three things that help to explain this:

1. Student skills in reading math (and math textbooks) lag behind the math content they are expected to learn.
2. Many math textbooks are not written in a manner that readily facilitates students learning the math by reading the book.
3. The "oral tradition" instruction techniques used in math teaching continue to dominate in precollege math instruction. Students get used to and come to expect this mode of instruction. Many experience a shock when their college math teachers expect them to read the textbook.

> Reading across the curriculum and learning by reading are a cornerstone of our education system. Unfortunately, our current math education system is quite weak in helping students learn to read math and learn math by reading.

Nowadays, essentially all students learn to make use of the Web. However, relatively few students learn to make use of the Web to access math-related materials. They learn to access sites specified by their teachers. Few learn to use the Web to explore math-related materials beyond what the teacher requires. Few learn to make use of the software designed to help represent and solve math problems. For example, are you and your tutees familiar with Wolfram Alpha (http://www.wolframalpha.com/)? (Also see http://iae-pedia.org/Free_Math_Software.)

Your authors believe this weakness in information retrieval and making use of Web-based resources represents a major flaw in our math education system. The combination of not developing appropriate levels of skill in reading math and not learning how to access math-related materials that they need or want to know about represents a major impediment to being a life long learner of math.

Project-Based Learning

> "I hear and I forget. I see and I remember. I do and I understand." (Confucius; Chinese thinker and social philosopher, whose teachings and philosophy have deeply influenced Chinese, Korean, Japanese, Taiwanese and Vietnamese thought and life; 551 BC–479 BC.)

Project-Based Learning (PBL) is an individual or group activity that goes on over a period of time, resulting in a product, presentation, or performance. It typically has a time line and milestones, and considerable formative evaluation as the project proceeds. See http://iae-pedia.org/Project-Based_Learning for an overview of Project-Based Learning. See Moursund (2008d) for a discussion of math Project-Based Learning.

Project-Based Learning and Problem-Based Learning share the same PBL initials. As teaching methodologies, they overlap. However, Project-Based Learning is much more learner centered.

1 Students have a significant voice in selecting the content areas and nature of the projects that they do. As compared with traditional teaching methodologies, Project-Based Learning gives increased power to students and makes them more responsible for their own learning.

2. Project-Based Learning has a considerable focus on students' understanding what it is they are doing, why it is important, and how they will be assessed. Indeed, students may help to set some of the goals over which they will be assessed and help to develop assessment rubrics.

3. The learner-centered characteristics of Project-Based Learning contribute to learner motivation and active engagement. A high level of intrinsic motivation and active engagement are essential to the success of Project-Based Learning.

Project-Based Learning incorporates the ideas of students learning from each other and students learning to work together as a team. These are both real-world skills. However, this mode of instruction presents some major challenges to teachers:

1. Content. Even if all of the projects being done by students focus within a restricted area (such as a particular math topic, historical event, or mathematician), each team is likely to be studying different parts of the area. This may not fit in well with preparing students for a high stakes state or national test.

2. Instruction. Students benefit from explicit teacher-driven instruction on how to function as a team. In facilitating teams working on their individual projects, the teacher will

detect times when it is appropriate to call the whole class together to present general information relevant to the whole class. This requires teachers to think on their feet and create an appropriate intervention on the fly.

3. Assessment. Project-Based Learning does not lend itself to students taking a uniform final exam. Teachers are also faced by the challenge of determining the contribution of each individual students working on a team, the progress each makes in learning how to be a collaborative and productive team member.

Distance Education and Computer-Assisted Learning

Early distance education—correspondence courses making use of postal delivery systems—dates back to the 1700s. In the absence of formal face-to-face schools, students could learn by reading—often with help from siblings and parents.

Computer-assisted learning can be thought of as a type of interactive book. Nowadays, this interactive book contains print, audio, and video content. Such a book may be wholly self-contained (it can even be a battery powered handheld device) or its content may be delivered through a telecommunications system such as the Internet.

Recent years have seen a merger of distance education and computer-assisted learning. We still have considerable distance education that consists of students viewing video recordings of lectures. However, contrast this with use of the huge and growing library of documentary videos and other videos of substantial educational value. Further, contrast this with the integration of lecture content and educational video content into well-organized interactive computer-assisted learning lessons.

We know that feedback is essential in learning. I thoroughly enjoyed watching my youngest grandchild learn to read out loud. As he read a sentence or worked to decode a word, he provided feedback to himself because he was already fluent in his natural spoken language. If a word or sentence did not make sense to him, he could often puzzle out his reading errors. Of course, much of this early learning occurred while sitting in the lap of a helpful adult, so there was a second feedback source.

The same type of situation existed as he learned to play computer games. Trial and error—with self-assessment and self-feedback—are a powerful aid to learning. Computerized games also enforce the rules, provide contextual instruction (instruction relevant to what is currently going on in the game), and provide an audiovisual environment that a game player uses to provide self-assessment and self-feedback.

Both human teachers and computer systems can provide certain types of feedback to students. Schools often combine human-based and computer-based instruction in hybrid courses. At the college level, for example, a course that traditionally has four class meetings a week may have just two meeting a week along with a substantial amount of computer-delivered instruction that occurs outside of these two class meetings. Students get the benefit of the computer doing what it can do best and the human teacher doing what he or she can do best.

Modern Highly Interactive Intelligent Computer-Assisted Learning systems (HIICAL) do all of this and more. Such a system contains the curriculum content to be taught, the instruction, and the assessment. It has built-in formative assessment with immediate feedback, and it can provide contextual instruction. It can provide links to additional online resources.

Distance education based HIICAL is a powerful change agent in our current educational system. Some school districts and state education system are beginning to require all of their students to experience some form of distance education. As HIICAL become more available, it will greatly change the roles of teachers.

Highly Interactive Intelligent Computer-Assisted Learning (HIICAL) is a very powerful change agent in education.

Final Remarks

A lesson plan includes information about curriculum content, instructional processes, and assessment. These three components are thoroughly interrelated and are often intertwined in a math lesson.

We are used to the idea of the traditional teacher-centered math teaching. Now, however, some completely self-contained HIICAL is available. HIICAL contains the content, instruction, and assessment. It is clear that we are in the initial phase of a major trend that will strongly impact our educational system. Moreover, we need to keep in mind that the medium and the message are intertwined. The computer system being used to teach math can also be used to solve math problems—and vice versa.

End of Chapter Activities

1. Think about your own precollege math education. To what extent can it be described as being based on an oral tradition teaching methodology? Reflect on your current skills in learning math by reading math books.
2. Reflect on your student experiences in Project-Based Learning. From your personal point of view, what are the strengths and weaknesses of Project-Based Learning? If you have experience Project-Based Learning or Problem-Based Learning in math, include these in your reflections.
3. Reflect on your personal experiences in use of computers as an aid to teaching. What are your relative strengths and weaknesses in using such aids as you teach a classroom full of students?
4. As a future or current teacher of math, what do you think and feel about potential changes that seem likely to occur as HIICAL comes into much more common use in math education?

Chapter 7: A "Full Blown" Math Lesson Plan Template

"If you don't know where you are going, you are likely to end up somewhere else." (Lawrence J. Peter, of "Peter's Principles" fame.

Introduction

Chapter 1 contains the following diagram:

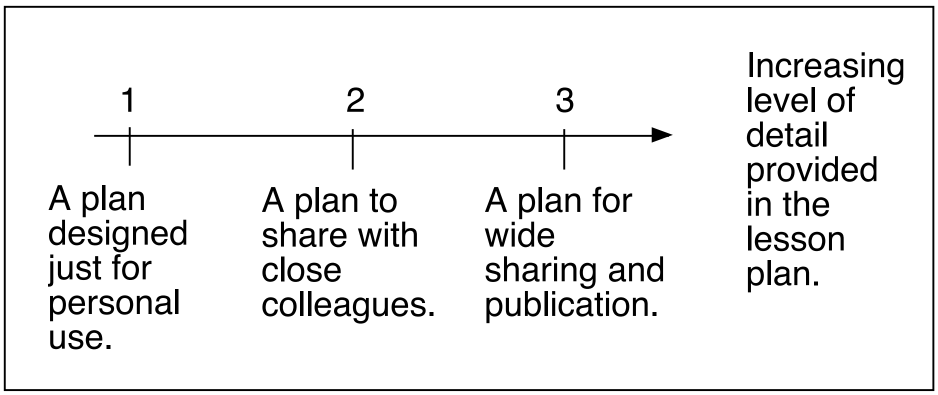


Figure 7.1. Types (levels) of lesson plans.

1. A personal lesson plan is a personal aid to memory that takes into consideration your expertise (teaching and subject area knowledge, skills, and experience). It's often quite short—sometimes just a brief list of topics to be covered or ideas to be discussed. For example:

    Give each student about 30 square tiles. The general goal is to explore forming connected geometric shapes that can be made from square tiles. Here, "connected" means that every tile in the geometric shape has at least one edge in common with another tile. Some of the shapes that can be formed have special names such as rectangle and square. Some are shaped like letters, such as an L. What letters can one make? What digits can one make? Figure out areas and perimeters of the connected figures. It is easy to see how to make different rectangles with areas 1, 2, 3, 4, 5, and so on. Can one make squares with areas 1, 2, 3, 4, 5, and so on? Why, or why not? Find examples of differently shaped rectangles that have the same area.

2. A collegial lesson plan is designed for a limited, special audience such as your colleagues, a substitute teacher, or a supervisor such as a principal. It contains considerably more detail than the first category. It is designed to communicate with people who are familiar with the school and curriculum of the lesson plan writer.

3. A (high quality) publishable lesson plan is much more detailed than a collegial lesson plan and is intended for use by a wide, diverse audience. It is designed to communicate with people who have no specific knowledge of the lesson plan writer's school, school

district, and state. It is especially useful to preservice teachers, to substitute teachers in unfamiliar situations, and to workshop presenters seeking to elicit in-depth discussion.

This current chapter presents a template for a somewhat traditional teacher-centered Level 3 math lesson plan. It includes some discussion that builds on Information and Communication Technology ideas from the previous chapter

A "Full Blown Traditional" (Type 3) Math Lesson Plan Template

This section contains a Level 3 general-purpose template for math lesson plans. It is a template for lesson plans to be used in teaching preservice and inservice teachers.

As you develop a lesson plan or prepare to teach from a lesson plan, think about the teacher prerequisite knowledge and skills needed to do a good job of teaching the lesson.

Before you teach a math lesson, do a self-assessment to determine if you have the needed math content knowledge, the general pedagogical knowledge, and the math pedagogical content knowledge. If you detect possible weaknesses, spend time better preparing yourself to teach the lesson, and spend time thinking about what you will learn as you teach the lesson. (See item 10 in the list given below.)

In addition, think about what you bring to the lesson plan and its implementation that is unique to you. Some teachers have what I call a "signature trait." In your teaching of math, what distinguishes you from other teachers? How do you personalize your math teaching? What do you do that a video of a good math lecturer cannot do

1. Title and short summary—like a section title in a book chapter (lesson plan) or a chapter title (unit plan). The title of a math lesson plan or unit should communicate purpose to the teacher and to students. It serves in part as an advance organizer. The short summary is part of the advance organizer for the lesson plan, and includes the expected time (length) of the lesson. t should include a statement of how the lesson or unit serves to empower students.
2. Intended audience and alignment with Standards. Categorization by: subject or course area; grade level; general math topic being taught; length; and so on. A listing of the math standards (state, province, national, etc.) being addressed. Categorization schemes are especially useful in a computer database of lessons, allowing users quickly to find lesson plans to fit their specific needs.
3. Prerequisites—a critical component in math lesson planning and teaching. How will you check to see if your students have the necessary prerequisite knowledge and skills? Math teachers and their students face the difficulty that a significant proportion of the class may not meet the prerequisites. Such students are not apt to learn the new material very well, and the lack of success will likely add to "I can't do math" and "I hate math" attitudes. Some of your students will probably not have the necessary prerequisite knowledge and skills. What do you plan to do to deal with this situation?
4. Accommodations—special provisions needed for students with documented exceptionalities and other students with math learning and math understanding differences from "average" students. This ties in closely with how to deal with students who clearly lack needed prerequisite math knowledge and skills. However, you also

need to plan an accommodation for students who are considerably more mathematically advanced than average and will be bored by your lesson's content.

5. Learning objectives. Teachers of teachers often stress the need for stating learning objectives precisely. They often use the expression *measureable behavioral objectives* (measureable results based on agreed-upon goals and objectives; see http://www.adprima.com/objectives.htm) . Some additional important aspects of the earning objectives section of a math lesson or unit of study are:

   a. Math expertise. Each lesson and unit of study needs to maintain and improve each student's overall level of math expertise. Math expertise combines math content knowledge and math maturity. It is important that students understand the idea of math expertise, how it grows through study, practice, and use, and how it decreases through lack of use (forgetting). Students need to learn to take personal responsibility for their levels of expertise. Every lesson should include an emphasis on self-assessment, self-responsibility, sense-making, and problem solving. Problem solving and proof are closely related topics; problem solving should be in ways that lay the foundations learning about proofs in math. Informal and/or proof-like presentations and discussions should be part of every unit of study.

   b. Math vocabulary, notation, and modeling. Keep in mind that math notation, vocabulary, and modeling tend to have a high level of abstraction. Math modeling is a process of extracting a "pure" math problem from a problem situation. This extraction or modeling process is a very important aspect of learning and understanding math. It is a challenge to teachers and to students.

   c. Lower order and higher-order. Make a clear distinction between lower-order and higher-order knowledge and skills. Both are essential to problem solving, and it is important for students to be learning and making use of both lower-order and higher-order aspects of problem solving in an integrated, everyday fashion. Note, of course, lower-order and higher-order are dependent on the math cognitive developmental level and math maturity of your students. Higher-order pushes the envelope—it helps students to increase their level of math development and math maturity. This ties in closely with (a) and (b) above.

   d. Transfer of learning. Each unit of study should include specific instruction on transfer of learning. A unit of study is long enough so that students can learn a strategy, or significantly increase their knowledge and understanding of a strategy, and gain increased skill in high-road transfer of this learning to problem solving across the curriculum.

   e. Communication in math. Part of this is students gaining skill in communicating with themselves—mental sense-making. Pay special attention to students learning how to read math well enough so that they can learn math by reading math. How will your lesson help students improve their math communication skills?

   f. Computational Thinking. Keep in mind the steadily growing importance of Computational Thinking in math and in other disciplines. Stress roles of ICT and a student's brain/mind in computational thinking. Help students learn the capabilities and limitations of brain/mind versus calculators and computers in representing and working to solve math problems. Stress how math is used to develop math models of problem situations to be explored and possibly solved in each discipline. Math is of growing importance in many disciplines because of its role in computational thinking and in using math models to represent and help solve the problems in these disciplines.

6. Materials and resources—These include reading material, assignment sheets, worksheets, tools, equipment, CDs, DVDs, videotapes, and so on. You may need to begin the acquisition process well in advance of teaching a lesson, and it may be that some of the resources are available online. If your lesson depends on use of calculators, computers, presentation media, and/or online materials, what is your backup plan if there is an equipment failure?
7. Instructional plan—This is usually considered to be the heart of a lesson plan. It provides instructions to the teacher to follow during the lesson. It may include details on questions to be asked during the presentation to students. If the lesson plan includes dividing students into discussion groups or work groups, the lesson plan may include details for the grouping process and instructions to be given to the groups.
   a. A carefully done math lesson plan includes a discussion of math pedagogical content knowledge (PCK) that has been found useful in helping students learn the topic.
   b. If students are going to be making use of math manipulatives, calculators, computers, and other ICT learning aids, pay special attention to the general pedagogical requirements and the PCK requirements of dealing with a large number of students. The cognitive and organizational load on a teacher dealing with a one-on-one computer situation can be rather overwhelming.
8. Assessment options—A teacher needs to deal with three general categories of assessment: formative, summative, and long-term residual impact. Students need to learn to do self-assessment and to provide formative assessment (evaluation during the process to aid progress) and perhaps summative assessment feedback (passing judgment on the final result) to each other. A rubric, perhaps jointly developed by the teacher and students, can be a useful aid to helping students take increased responsibility for their own learning.
9. Extensions—These may be designed to create a longer or more intense lesson. For example, if the class is able to cover the material in a lesson much faster than expected, extensions may prove helpful. Extensions may also be useful in various parts of a lesson where the teacher (and class) decide as the lesson is being taught that more time is needed on a particular topic.
10. References—The reference list might include other materials of possible interest to people reading the lesson plan or to students who are being taught using the lesson plan. Emphasize readily available materials, such as those available (free) on the Web.
11. Teacher learning on the job—View each math lesson and unit of instruction as an opportunity to increase your knowledge and skills in math content, math pedagogy, and general pedagogy. Set specific learning goals and objectives for yourself. After teaching a lesson or a unit of study, reflect on what you have learned. Add some notes to your

lesson plan that reflect your increased knowledge and skills, and that provide a sense of direction for focusing your learning the next time you teach the lesson or unit.

View lesson planning and teaching as a type of inservice self-education. After planning and teaching a lesson, reflect on what you have learned and update your lesson plan to reflect your new insights.

Hybrid Teaching Environments

The term *hybrid teaching environment* usually means a situation in which students spend a considerable amount of time in an online learning environment and also spend time in formal class meetings. The number of hours of class meetings might be half of those for a non-hybrid course.

I find it useful to consider a somewhat more general definition of hybrid. Suppose that a teacher in a "regular" course makes extensive use of videos. I consider the following to be an example of a hybrid lesson. The teacher's implementation plan for such a 50-minute video-based lesson might consist of:

1. Get the class started and introduce a video to be shown. (5 minutes)
2. Show a video. (13 minutes)
3. Have students do small group discussion to identify the most important ideas in the video and how these ideas relate to the course. Circulate among the groups, listening for key ideas that are being discussed. (15 minutes).
4. Do a whole class discussion sharing and summarizing the ideas discussed in the small groups, with a focus on emphasizing "big ideas" that were and/or were not discussed in the small groups. (15 minutes)
5. Closure. (2 minutes)

In this example, about a fourth of the class time is spent viewing the video, and a little more than a quarter of time class time is spent in small group discussion.

The teacher may need to spend considerable preparation time in advance of teaching such a lesson. This includes viewing and reviewing videos, preparing an advance organizer to be used to get the class started, preparing questions to facilitate small group discussions, and deciding on the big ideas to be covered or reviewed in the whole class discussion. The teacher makes mental or written notes during the student discussion time.

There are many good math-related videos. For example, see http://iae-pedia.org/Math_Education_Free_Videos.

In addition, there are good projections systems that can display output from a calculator. With such equipment, a math teacher can interact with the class and present examples of calculator and computer-assisted math problem solving. While it is a stretch to call this a hybrid model of teaching, it is an excellent example of making use of calculator and computer technology in a math classroom.

Especially in higher education, we are seeing a strong trend toward hybrid courses. The in-class use of videos provides an example of such hybrid teaching. This is being facilitated by computer storage and computer projectors.

Final Remarks

Developing a full-blown lesson plan and/or adapting such a plan that has been developed by others can be a lot of work. With practice, however, it becomes a relatively easy task that can be completed fairly quickly. Also, as you continue your teacher career, you can accumulate a lot of lesson plans and related materials (for example, handouts and quizzes) that you have previously used. These can be updated when the need arises.

End of Chapter Activities

Select a math topic that you teach or are preparing to teach. With that topic in mind:

1. Reflect on the prerequisite knowledge and skills the topic assumes. What are some good ways to quickly determine if most of the students in your class have the prerequisite math knowledge and skills?
2. Explain why the topic is important. (How will you handle a student question, "Why do we have to learn this?")
3. What aspects of the math topic you have selected seem to you to be lower order and what aspects seem to you to be higher order? What aspects do you feel will be fairly easy for most students, and what aspects do you feel will be fairly difficult? Why?
4. Reflect on transfer of learning of this topic. Can you give examples of possible transfer to outside of school situations and to situations your students currently face in courses they are taking?
5. Reflect on your "signature traits" as a current or future math teacher. How do these help to make you a successful and memorable teacher?

Chapter 8: Final Remarks and Closure

"Chance favors only the prepared mind." (Louis Pasture; French chemist and microbiologist; 1822–1895.)

This final chapter briefly introduces additional topics that are important in developing and implementing good math lesson plans.

Reading, Writing, and Mathing Brains

It took you considerable informal and formal education, time, and effort to develop your current skills in reading, writing, and mathing. (I like the word mathing, although it is not widely used. You might enjoy doing a Web search of this term.) This learning rewired various parts of your brain. I like to think of this process as developing a reading brain, a writing brain, and a mathing brain. Of course, the same idea applies to developing significant levels of knowledge and skills in other disciplines. Via music education one develops a music brain; through art education one develops an art brain.

Howard Gardner's theory of multiple intelligences argues that we have innate logical/mathematical abilities. Keith Devlin argues that the ability to learn to communicate in a natural language means that one can learn math (Devlin, 2000). As we nurture and develop our innate math abilities, we are enhancing (growing, building, changing) our math brain.

Reading and writing are powerful aids to human thinking and problem solving. In essence, they provide an auxiliary brain. Math is a language. As we learn to read and write math, we are improving and learning to make broader use of our math auxiliary brain.

History going back even further than the first development of reading and writing indicates that humans developed external aids to their math brain. Notches in a stick, scratches on a bone, marks in the sand (a sand abacus), and pebbles in a pouch (in one-to-one correspondence with the goats in one's herd) are examples of such aids.

Over thousands of years, the reading, writing, and external aids to one's math brain have steadily improved. Over the past 70 years the electronic digital computer has been developed and substantially improved year after year. A computer brain and a human brain are quite different. A computer brain can do many things that a human brain cannot do, and vice versa. Taken together, these two types of brains can solve many problems and accomplish many tasks—in math and many other disciplines—that neither can do alone (Moursund, 2008a).

Ongoing research in cognitive neuroscience is helping to build our underlying foundation for significant improvements in education.

Use of Games in Math Education

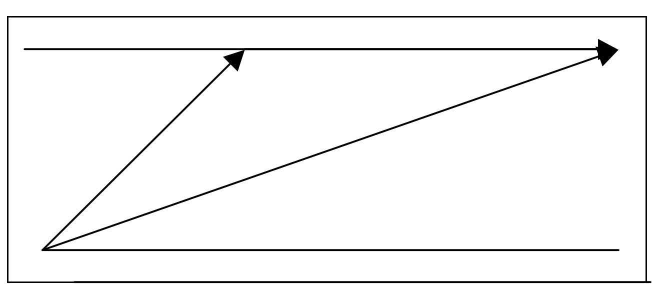
Bob Albrecht and I have written a book on the use of games in math education (Moursund & Albrecht, 2011a). Most people can remember back to their childhood and games that they played that involved numbers, rolling dice, spinning spinners, use of play money, playing cards, and so on. Monopoly and other games were an important part of my childhood.

The idea is simple enough. Use games to create a fun environment in which a person is intrinsically motivated to participate. There may also be extrinsic motivation, such as the social interaction of being with friends. Make sure that playing the game involves activities reasonably similar to what we want a child (or, student of any age) to learn, because we want transfer of learning to occur. There is substantial research supporting the successful use of games in education. Some of this is discussed in Moursund & Albrecht (2011a).

Let me carry this line of thinking a little further. For me and many other people, math is fun. For me, math is a type of game involving challenging mental tasks and intrinsic motivation. I pose math problems to myself and then I think about them and try to solve them. I find brain teasers on the Web, and I do metacognition as I let these puzzles mess with my brain. I read the bridge column in my local newspaper, and try to figure out both the bidding and the play of the cards.

I view the world through "math-colored" glasses. For example, recently at dinner with some friends, a waiter was serving coffee refills. The waiter asked one person if she wanted just a half-cup. She replied no, she wanted a full cup—because the coffee cools off too rapidly if she has only a half-cup. Hmm, I said to myself. Is she correct? What math and science do I know that would support or disprove her assertion? My math brain told me to think of the range of possible situations. (In math terms, think of the limiting case.) Suppose there is just a tiny bit of coffee in the cup. Will it cool off faster that a full cup—and why? Does the shape of the cup matter? Does the thickness of the cup matter? For me, this type of problem recognition or problem posing, and then trying to solve the problem, is a fun game.

Here is another example. Before I retired, I would walk from home to my University of Oregon office on nice days. Along the way I had to cross a number of streets. Sometimes I crossed in the middle of a block, but not exactly perpendicular to the street. Hmm. Does crossing at a 45-degree angle save me the most time? Why not cross at a 30-degree angle or a 60-degtree angle?



A good math teacher can help make math fun. A student can learn that math is fun.

Humor

My recent Web search for math jokes produced nearly 2 million hits. (I also got a lot of his using the search expressions *math cartoons* and *math comics*.) Here are a few of the jokes:

- Why did the boy eat his *math* homework? Because his teacher told him it was a piece of cake!
- There are three kinds of mathematicians: those who can count and those who can't.
- My geometry teacher was sometimes acute, and sometimes obtuse, but always right.

See Mary Kay Morrison's article about humor in education (Morrison, 2012). . The article lists 10 reasons for using humor in education. Here are two items from her list:

- Humor captures and retains attention. Laughter and surprise can hook even the most reluctant student. "Emotion drives attention and attention drives memory, learning, problem solving, and behavior." The brain cannot learn if it is not attending. Humor generates something unexpected, which alerts the attentional center of the brain and increases the likelihood of information recall. It can be integrated into all aspects of the learning process as described in the *Educators Tackle Box in Using Humor to Maximize Learning* (Morrison, 2008). The purposeful use of humor is a skill that can be practiced and enhanced. A favorite follow-up strategy is to invite the students to read a section of the lesson and create a joke or riddle about that segment. Some of these can be used in the actual test for the chapter. Lost In Thought–It's Unfamiliar Territory!
- Humor neutralizes stress. Humor will decrease depression, loneliness and anger. The contagious nature of laughter is caused by mirror neurons—brain cells that become active when an organism is watching an expression or goal-directed behavior that they themselves can perform. If you see someone laughing, even if you don't know the reason for the laughter, you will probably laugh anyway. The imitative behavior is due to mirror neurons being stimulated. Stress levels have been increasing for both students and teacher. Laughter is contagious. Catch it! Spread it! He Who Laughs–Lasts!

Math Riddles and Brain Teasers

My recent Web search of "math riddles" OR "brain teasers" produced a huge number of hits. Here is a riddle named *Three Math Teachers at a Hotel.*

> Three math teachers rent a hotel room for the night. When they get to the hotel they pay the $30 fee, and then go up to their room. Soon the bellhop brings up their bags and gives the math teachers back $5 because the hotel was having a special discount that weekend. So the three math teachers decide to each keep one dollar and to give the bellhop a $2 tip.
>
> However, when they sat down to tally up their expenses for the weekend they could not explain the following details. Each one of them had originally paid $10 (towards the initial $30), and then each got back $1 which meant that they each paid $9. Then they gave the bellhop a $2 tip. However, **3 x $9 + $2 = $29.** The math teachers couldn't figure out what happened to the other dollar. After all, the three paid out $30 but could only account for $29.

Can you determine what happened?

Here is a "classic" proof that 2 = 1 that is a brain teaser. Explain what is wrong with this proof.

Suppose that $x = y$. Then

$2x - x = 2y - y$. This implies

$2x - 2y = x - y$. We can rewrite this as

$2(x - y) = (x - y)$. We now divide each side by $(x - y)$ and we get

$2 = 1$.

Peer Tutoring in Math

> "When you teach, you learn twice." (Seneca; Roman philosopher and advocate of cooperative learning; 4 BC–65 AD.)

Chapter 10 of Moursund and Albrecht (2011b) focuses on peer tutoring (paired learning). In peer tutoring:

1. The tutor and tutee taking advantage of their shared learning experiences and their understandings of challenges they have faced and are facing in their informal and formal educational systems.
2. The tutee and the tutor each gain knowledge and experience through working together.

There has been substantial research on the effectiveness of peer tutoring. Peer tutoring can be thought of as a type of cooperative learning. Quoting from Alfie Kohn's 1993 book *Punishment by Rewards:*

> One of the most exciting developments in modern education goes by the name of cooperative (or collaborative) learning and has children working in pairs or small groups. An impressive collection of studies has shown that participation in well-functioning cooperative groups leads students to feel more positive about themselves, about each other, and about the subject they're studying. Students also learn more effectively on a variety of measures when they can learn with each other instead of against each other or apart from each other. Cooperative learning works with kindergartners and graduate students, with students who struggle to understand and students who pick things up instantly; it works for math and science, language skills and social studies, fine arts and foreign languages.

Students Taking Increased Responsibility for Their Own Learning

There are many things a teacher can do to help students take increased responsibility for their own math learning. Here are three examples:

1. Help students learn to read their math books. Create a teaching and learning environment that routinely supports this endeavor. Give some assignments in which students must use the Web or other resources to locate, read with understanding, and use the math-related information they retrieve. Experiment with open book and open computer tests.

2. Help students learn to self-assess their math knowledge, skills, and understanding. Provide them with computer-based and other self-assessment instruments. Help students learn how to check their answers for reasonableness or exact correctness. Don't make students do a lot of busy work (such as drill and practice) on procedures that they know they have already mastered. Be aware that some students achieve mastery much more rapidly than others.
3. Make use of individual and small group math project-based learning.

Quoting from Richardson (2012):

> Between adaptive software that can present and assess mastery of content, video games and simulations that can engage kids on a different level, and mobile technologies and online environments that allow learning to happen on demand, we need to fundamentally rethink what we do in the classroom with kids ….
>
> That rethinking revolves around a fundamental question: When we have an easy connection to the people and resources we need to learn whatever and whenever we want, what fundamental changes need to happen in schools to provide students with the skills and experiences they need to do this type of learning well? Or, to put it more succinctly, **are we preparing students to learn without us**? How can we shift curriculum and pedagogy to more effectively help students form and answer their own questions, develop patience with uncertainty and ambiguity, appreciate and learn from failure, and develop the ability to go deeply into the subjects about which they have a passion to learn? [Bold added for emphasis.]

Examples of Good and Not-so-good Math Lesson Plans

This book notes that thee are "oodles" of math lesson plans available on the Web. However, the book does not contain specific detailed examples of lesson plan.

The Website http://iae-pedia.org/Sources_of_Math_Lesson_Plans contains links to a large number of math lesson plans that are available on the Web.

Final Remarks

Teaching is both an art and a science. Whether you are a preservice or an inservice teacher, you know some of your strengths and some of your weaknesses as a teacher of math. I hope that this book has helped you to better understand what constitutes a good math lesson plan and ways in which to improve a math lesson plan.

A good math lesson plan is only part of what it takes to be a good, effective math teacher. Think about some of the ways you know to get students intrinsically motivated to learn and do math. Success in increasing intrinsic motivation and personal student commitment depends on your personal characteristics and human-to-human interactive skills. These are not captured in a lesson plan—they are captured in your implementation of a lesson plan and your overall interactions with your class and individual students.

Math is a human endeavor. Learning math is a human endeavor. Teaching by a human teacher is a human endeavor. A good math teacher is a powerful aid to student learning.

Teaching is both an art and a science. The science of teaching and learning is steadily being improved by research and by use of technology.

End of Chapter Activities

1. Look back at section in Chapter 1 titled **Math Lesson Planning: It's Easy—Right?** Reflect on some things that you have learned by reading this book that you feel are relevant to good math teaching and that are not captured in the fictitious students' insights into math teaching.
2. Select one or two ideas from this book that could help you become a better math teacher. Get them clearly in mind, and then reflect on how you could (will!!!) go about implementing them.
3. Think about sharing ideas from this book with colleague. What would you say to encourage a colleague to read the book? What would you say to discourage a colleague from reading the book?

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